Lecture 4

projectivise

a slope α is an element of $PH_1(\partial M;\mathbb{R})$

where $\alpha \in H_1(\partial M;\mathbb{R}) \setminus \{0\}$

if M is a knot complement, then $H_1(\partial M;\mathbb{Z}) \cong \pi_1(\partial M)$

has generators $\{\mu, \lambda\}$

any $[\alpha]$ with $\alpha \in \pi_1(M)$ can be written $\mu^p\lambda^q \longleftrightarrow p/q$

We'll address $\pi_1(M(p/q))$ where

$$M(p/q) = M \cup_f (S^1 \times D^2) \quad \text{Dehn fill}$$

and therefore

$$\pi_1(M(p/q)) = \pi_1(M) / \langle\langle \mu^p\lambda^q \rangle\rangle$$

is this left orderable or not?

Here, we get homology $\mathbb{Z}/p\mathbb{Z}$, and we get irreducible almost always

We should get:

Conj: M a knot complement
if $\exists\, r \in \mathbb{Q}$ $r > 0$ s.t. $\pi_1(M(r))$ is not LO
then $\pi_1(M(p/q))$ is LO $\iff$ $p/q \in (-\infty, 2g(K)-1)$

(special case of L-space conjecture)

Take away: Sometimes LO
sometimes not
need tools for each

To show Dehn fillings are LO:

(i) somehow create $\rho_s : \pi_1(M) \to Homeo_+(S^1)$ w/ nonabelian image
(ii) given $\alpha = \mu^p \lambda^q$ find s s.t. ρ_s factors

$$\begin{tikzcd} \pi_1(M) \arrow[rr, "\rho_s"] \arrow[dr] & & Homeo_+(S^1) \\ & \pi_1(M(\alpha)) \arrow[ur] & \end{tikzcd}$$

(iii) conclude $\pi_1(M(\alpha))$ is CO, adjust the CO f so that
$[f] = id \in H^2$

Lemma:

Every non-abelian CO group is infinite

Pf: if f is a circular order on a finite group G

consider

$$0 \to \mathbb{Z} \to \tilde{G}_f \to G \to 1 \qquad \text{from last time}$$

$$\uparrow LO$$

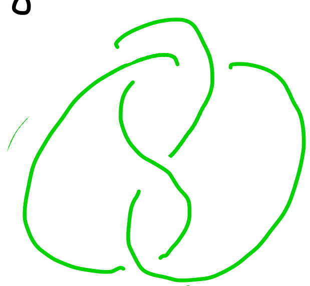
$\Rightarrow \tilde{G}_f$ is torsion free, virtually cyclic

$\Rightarrow \tilde{G}_f$ is cyclic (Stallings)

$\Rightarrow G$ is cyclic

(so where last time needed ∞ image but now only require non-abelian)

Start with figure 8



calculate

$$G = \langle x, y \mid wx = yw \rangle$$

$$w = xy^{-1}x^{-1}y$$

here x represents μ

$$\lambda = y x^{-1} y^{-1} x^{2} y^{-1} x^{-1} y$$

we can find reps $\rho : G \to PSL(2,\mathbb{C})$

by solving the Riley polynomial

in fact, real solutions can give rep^n s

into $PSL(2,\mathbb{R})$

for $s \geq \frac{1+\sqrt{5}}{2}$ and $t = \frac{1+\sqrt{(s-s^{-1})^4 + 2(s-s^{-1})^2 - 3}}{2(s-s^{-1})} \in \mathbb{R}$

we get $\rho_s : G \to PSL(2,\mathbb{R})$

$$\rho_s(x) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$$

$$\rho_s(y) = \begin{pmatrix} \frac{s+s^{-1}}{2} + t & \frac{s-s^{-1}}{2} + t \\ \frac{s-s^{-1}}{2} - t & \frac{s+s^{-1}}{2} + t \end{pmatrix}$$

find all $p/q \in \mathbb{Q}$ st. $\exists\, s$ with $\rho_s(\mu^p \lambda^q) = \pm I$

Eventually arrive at $[0,4)$ is what works

i.e. $p/q \in [0,4) \Rightarrow \exists s$ s.t. ρ_s factors through $\pi_1(M(p/q))$

$\Rightarrow \pi_1(M(p/q))$ is CO

all surgeries irreducible $\Rightarrow \exists$ a subgroup $H \leq \pi_1(M(p/q))$ that is LO with $\pi_1(M(p/q))/H$ cyclic

$\Rightarrow [\pi_1(M(p/q)), \pi_1(M(p/q))] \leq H$ and so it is LO

Remark:

(i) Can produce CO f of $\pi_1(M(p/q))$ with $[f] = id \in H^2 \Rightarrow \pi_1(M(p/q))$ is LO

(ii) this gives LO for all $p/q \in (-4,4)$

(iii) other techniques use fol$^{\underline{n}}$s or flows to get ρ_s

note all surgeries on fig 8 LO but only recently done

Techniques for obstructing $\pi_1(M(P_q))$ LO

for a LO group G, recall

$$LO(G) = \{P \subset G \mid P \text{ a pos cone}\}$$

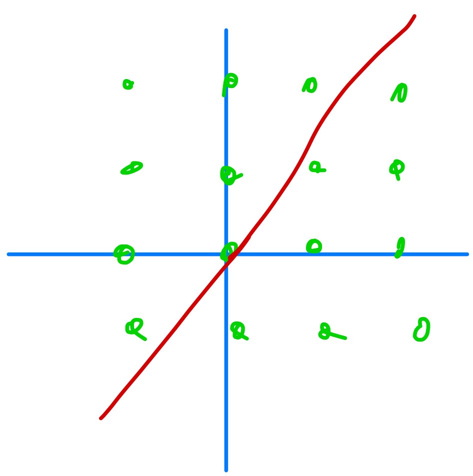
space of LO's

whenever $H \leq G$ there's a restriction map

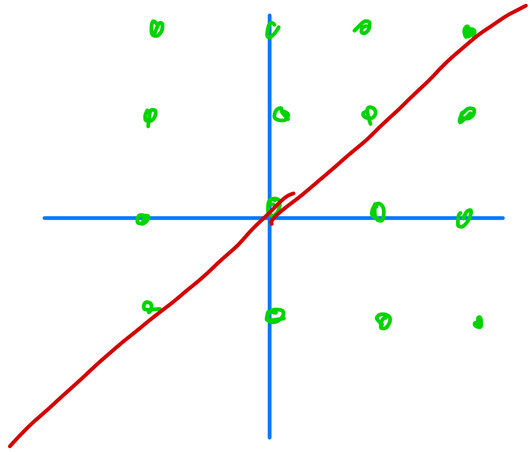
$$r : LO(G) \to LO(H)$$
$$P \longmapsto P \cap H$$

example: the space $LO(\mathbb{Z}^2)$

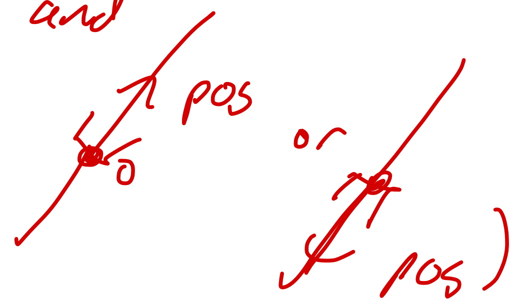
given $L \subset \mathbb{R}^2$ can make a LO of $\mathbb{Z}^2$



irrational slope
get 2 orderings
(above or below line)



rational slope
4 LO's
(above, below and



In fact, all orderings arise this way

given $P \in LO(\mathbb{Z}^2) \xrightarrow{\text{extend}} P' \in LO(\mathbb{Q}^2)$

$L(P) = \{ x \in \mathbb{R}^2 \mid$ every nbhd of x contains pts of P' and $(P')^{-1} \}$

So there is a map
$$LO(\mathbb{Z}^2) \longrightarrow \mathbb{RP}^1$$
$$P \longmapsto [L(P)]$$

So, if M is a knot complement, there is a map

$$s : LO(\pi_1(M)) \hookrightarrow LO(\pi_1(\partial M)) \longrightarrow S(M)$$

$$\pi_1(\partial M) = \mathbb{Z}^2, \qquad S(M) = \text{slopes on } \partial M$$

$$P \longmapsto P \cap \pi_1(\partial M) \longmapsto [L(P \cap \pi_1(\partial M))]$$

s is "the slope map"

<u>Prop:</u>

M knot comp

if $M(\alpha)$ is LO then $[\alpha]$ is in the image of s

<u>Pf:</u> use the short exact sequence

$$1 \longrightarrow \langle\langle \alpha \rangle\rangle \longrightarrow \pi_1(M) \longrightarrow \pi_1(M(\alpha)) \longrightarrow 0$$

LO — Lexicographic LO P — LO

then check $s(P) = [\alpha]$ ✓

Trick:

Prop: Suppose $p, q, r, s > 0$ and $p/q > r/s > 0$ and

M knot complement

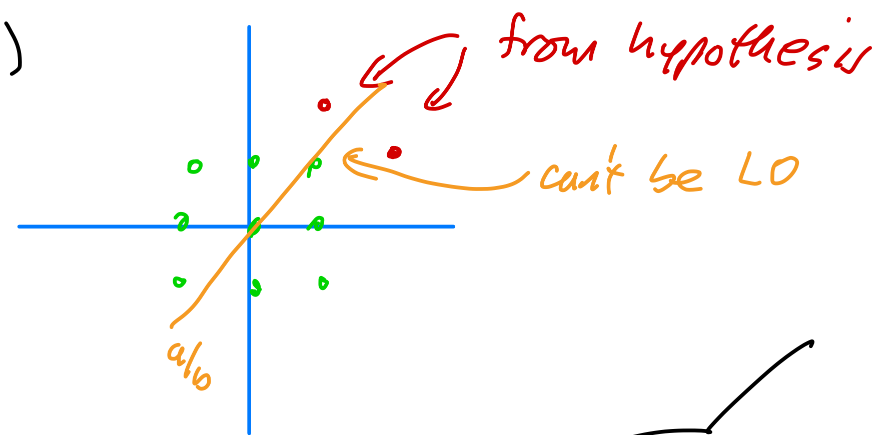
If every $P \in LO(\pi_1(M))$ satisfies

$$\mu^p \lambda^q \in P \Rightarrow \mu^r \lambda^s \in P$$

then $\forall\, a/b \in (r/s, p/q)$, then

$\pi_1(M(a/b))$ is <u>not</u> LO

"Pf": $\pi_1(\partial M)$



example: (p,q) torus knots

here, the group is $G = \langle a, b \mid a^p = b^q \rangle$

$$\mu = b^j a^i \qquad \begin{array}{l} pj + qi = 1 \\ p > i > 0 \\ 0 > j > -q \end{array}$$

$$\lambda = \mu^{-pq} a^p$$

assume $\mu^{pq}\lambda = a^p > id \Rightarrow a, b > id$

$$(\mu^{pq}\lambda)\mu^{-1} = a^p \cdot a^{-i} b^{-j} = a^{p-i} b^{-j} > id$$

$\Longrightarrow$ $\mu^{pq-1}\lambda$ to $\mu^{pq}\lambda$

have no LO fillings

Is it true that

$\pi_1(M(\alpha))$ is LO $\Leftrightarrow$ $[\alpha]$ is in the image of s?

No! the Dehornoy ordering (pos cone P_D) is a LO of $\langle a, b \mid a^3 = b^2 \rangle = B_3$ braid group it satisfies $s(P_D) = \mu$

can show all fillings in $(-\infty, 1)$ are LO

$(-\infty, 1) \subset \mathrm{Im}(s)$

$s : LO(\pi_1(M)) \longrightarrow S(M)$

so image contains $[-\infty, 1]$