Lecture 5

JSJ decomposition

M 3-manifold (irreducible)

$\exists$ collection $\mathcal{T}$ of tori in M s.t.

$M \setminus \mathcal{T} = M_1 \cup \dots \cup M_n$

each M_i is Seifert fibered or atoroidal

$\Rightarrow$ graph of groups $\left\{ \begin{array}{l} \pi_1(M) \text{ has vertex groups } \pi_1(M_i) \\ \qquad \text{edge groups } \mathbb{Z} \oplus \mathbb{Z} \end{array} \right.$ each LO

amalgams: A, G, H groups, injective homom

$$\phi_1 : A \to G, \quad \phi_2 : A \to H$$

set $S \subset G * H$

$$S = \{ \phi_1(a)\, \phi_2(a^{-1}) \mid a \in A \}$$

then $G *_{\phi_i} H = G * H / \langle\langle S \rangle\rangle$

example: twisted I-bundle over Klein bundle

consider $f(x,y,z) = (x+1, y, z)$

$g(x,y,z) = (-x, y+1, -z)$

$N = \left(\mathbb{R}^2 \times [-1/2, 1/2]\right) / K$

K = group gen by f, g

$\partial N = T^2$

observe that $fgf^{-1} = g^{-1}$ so

$\pi_1(N) = K = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$

$\pi_1(\partial N) = \langle x^2, y \rangle$

a LO fact about K:

it has only 4 LO's, all coming from

$1 \to \langle y \rangle \to K \to \mathbb{Z} \to 0$

take copies K_1, K_2 of K, $A = \mathbb{Z} \oplus \mathbb{Z}$

define $\phi_1 : A \to K_1$ $\quad \phi_2 : A \to K_2$

$\phi_1(0,1) = y_1 \quad \phi_2(0,1) = x_2^2$

$\phi_1(1,0) = x_1^2 \quad \phi_2(1,0) = y_2$

try to LO $K_1 *_{\phi_i} K_2$

eg supposing $1 < y_1 < x_1 < x_1^2$

$$\begin{array}{ccc} y_1 & & x_1^2 \\ \Updownarrow & & \Updownarrow \\ x_2^2 & < & y_2 \end{array}$$

this causes a problem, since any ordering of K_2 is lex., where y_2 is "infinitely small"

can be made concrete

Conclusion: It is not LO and π_1 of a 3-manifold

Recall $LO(G) = \{P \subset G \mid P \text{ a pos cone}\}$

this has a G action: $g \cdot P = gPg^{-1}$

call a subset $N \subset LO(G)$ <u>normal</u> if it is G-invariant

Th^m (Bludov - Glass):

Suppose A, G, H are groups with inj homom as above

$$\phi_1 : A \to G, \quad \phi_2 : A \to H$$

then $G *_{\phi_i} H$ is LO

$\iff$

$\exists$ normal families $N_1 \subset LO(G)$, $N_2 \subset LO(H)$ s.t.

$\forall P \in N_i$, $\exists\, Q \in N_j$ s.t. $\phi_i^{-1}(P) = \phi_j^{-1}(Q)$ for every $i, j \in \{1,2\}$

Corollary:

A, G, H as above

If A is infinite cyclic, then $G *_{\phi_i} H$ is LO if G, H are

Pf: $N_1 = LO(G)$

$N_2 = LO(H)$

If $P \in N_1$, then $\{\phi_1^{-1}(P), \phi_1^{-1}(P^{-1})\} = LO(A)$

$Q \in N_2$, then $\{\phi_2^{-1}(Q), \phi_2^{-1}(Q^{-1})\} = LO(A)$

so matching or "compatibility" is easy ✓

Th^{m} (C, Lidman, Watson):

M_1, M_2 3-manifolds, incompressible boundaries

$\phi : \partial M_1 \to \partial M_2$

$W = M_1 \cup_\phi M_2$

if $\exists\, \alpha$ s.t. $\pi_1(M_1(\alpha))$ and $\pi_1(M_2(\phi_*(\alpha)))$ are LO

and W is irreducible

then $\pi_1(W)$ is LO

Pf: let $G_i = \pi_1(M_i)$

$f_i : \mathbb{Z}\oplus\mathbb{Z} \to G_i$ inclusion of $\pi_1(\partial M_i)$

s.t. $\phi_* \circ f_1 = f_2$

let $q_1 : G_1 \to G_1/\langle\langle \alpha \rangle\rangle$ $\quad q_2 : G_2 \to G_2/\langle\langle \phi_*(\alpha) \rangle\rangle$

<u>now consider cases:</u> What are the possibilities for $\langle\langle \alpha \rangle\rangle \cap \pi_1(\partial M_1)$ and $\langle\langle \phi_*(\alpha) \rangle\rangle \cap \pi_1(\partial M_2)$?

could be $\underbrace{\mathbb{Z}\oplus n\mathbb{Z}}_{\cancel{\emptyset}}$, $\underset{\text{easy}}{\mathbb{Z}\oplus\mathbb{Z}}$, cyclic

gives torsion so not LO

if both cyclic, use

$\bar{\Phi} : q_1(\pi_1(\partial M_1)) \to q_2(\pi_1(\partial M_2))$

we get

$$\begin{tikzcd}
\mathbb{Z}\oplus\mathbb{Z} \arrow[r, "f_2"] \arrow[d, "f_1"'] & G_2 \arrow[d] \arrow[dr, bend left, "q_2"] & \\
G_1 \arrow[r] \arrow[dr, bend right, "q_2"'] & G_1 *_{\phi_2} G_2 \arrow[dr, dashed, "\exists \text{ homom}"] & G_2/\langle\langle \phi_*(\alpha) \rangle\rangle \arrow[d] \\
& G_1/\langle\langle \alpha \rangle\rangle \arrow[r] & G_1/\langle\langle \alpha \rangle\rangle *_{\bar{\Phi}_i} G_2/\langle\langle \phi_*(\alpha) \rangle\rangle
\end{tikzcd}$$

is LO by last result (cyclic amalgam)

because of LO quotient, $\pi_1(W)$ is LO

How strong is this?

Thm (C, L, W): completely handles integer homology sphere graph manifolds so they have LO π_1 (except S^3, $\Sigma(2,3,5)$)

Question: Are there examples where this is not enough?

Yes

example: This technique ~~fails~~ fails in general

i.e. $\exists M_1, M_2 \quad \phi: \partial M_1 \to \partial M_2$ s.t. $\pi_1(M_1 \cup_\phi M_2)$ is LO

but $\nexists\ \alpha$ s.t. $\pi_1(M_1(\alpha))$ is LO and

$\pi_1(M_2(\phi_*(\alpha)))$ is LO

the pieces $M_1 =$ trefoil complement

$\pi_1(M_1) = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$

$M_2 =$ twisted I-bundle over Klein bottle

$\pi_1(M_2) = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$

$\pi_1(\partial M_1) = \langle \underset{\mu}{\underset{\shortparallel}{\sigma_2}}, \lambda \rangle \qquad \lambda = \Delta^2 \sigma_2^{-6}$

$\Delta = \sigma_1\sigma_2\sigma_1$

$\pi_1(\partial M_2) = \langle y, x^2 \rangle$ this map is

$\phi(\sigma_2) = y^{-1}$

$\phi(\Delta^2) = y^{-1}x^2$

this is +4 surgery on fig 8

Prop: $\pi_1(W)$ cannot be left ordered by above thm

$(W = M_1 \cup_\phi M_2)$

Pf: only $\phi_*(\alpha) \in \langle y, x^2 \rangle$ that gives

$\pi_1(M_2(\phi_*(\alpha)))$ LO is

$\phi_*(\alpha) = y$

So on M_1 side, $\pi_1(M_1(\mu_1))$ would have to be LO

but its not ✓

Conclusion: need to build normal families

this can be done using Dehornoy ordering

In general, if we are trying to LO $G *_{\phi_i} H$

the "best case scenario" would be if

$\exists P \in G, Q \subset H$ s.t. $\phi_1^{-1}(P) = \phi_2^{-1}(Q)$

$? \Downarrow ?$

$G *_{\phi_2} H$ is LO

Conjecture: if $M_1, M_2, \phi: \partial M_1 \to \partial M_2$

then $\pi_1(M_1 \cup_\phi M_2)$ is LO

$\Leftarrow$

$\exists\, P \in \pi_1(M_1),\ Q \subset \pi_1(M_2)$

s.t. $\phi^{-1}(Q) = P$ (i.e. they agree)

<u>another version:</u>

$\pi_1(M)$ is LO

$\Longleftrightarrow$

$\phi_* \circ s_1(LO(\pi_1(M_1))) \cap s_2(LO(\pi_1(M_2))) = \emptyset$

here s_1, s_2 are slope maps

if you ask for a $P \subset \pi_1(M_1)$ s.t.

$s(P) \subset s(gPg^{-1})$

then we say $s(P)$ is order-detected.