Introduction to Heegaard Floer Homology I

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Lecture plan

1) Applications and formal properties of 3-mfd HF
2)-3) Sketch of construction of 3-mfd HF
4) Applications and formal properties of knot Floer homology
5) Sketch of construction of KHF; surgery formula

What can HF do for you?

Questions about 3-mfds

1) L-space conjecture
2) Thurston norm:

   given $\phi \in H_2(Y;\mathbb{Z})$ can ask:

   what is minimal genus of a surface $S \subset Y$ with $[S] = \phi$

3) Every 3-mfd is surgery on a link, what is the minimal # of components

   Is Y surgery on a knot?

Questions about 4-manifold that go between 3-mfds

1) Any Y is the boundary of some smooth 4-mfd X

is Y the boundary of a positive definite X?

" " negative " "

2) given a $\mathbb{Z}HS^3$ Y. Does Y bound an $\mathbb{Z}HB^4$?

always a topological $\mathbb{Z}HB^4$ that it bounds

(Freedman)

<u>Formal prop. of examples</u>

$\widehat{HF}(Y)$: this is a vector space over $\mathbb{F}$ $(= \mathbb{Z}/2\mathbb{Z})$

$$\widehat{HF}(Y) = \bigoplus_{\substack{\text{spin}^c \text{ strs} \\ s \text{ on } Y}} \widehat{HF}(Y; s)$$

$\longleftrightarrow$ $H^2(Y, \mathbb{Z}) = H_1(Y; \mathbb{Z})$

if s is a torsion spin^c structure, then

$\widehat{HF}(Y, s)$ has a grading by a coset of

$\mathbb{Z}$ in $\mathbb{Q}$

<u>Examples</u>: 1) $\widehat{HF}(S^3) = \mathbb{F}_0$ ← grading

2) $\widehat{HF}(\Sigma(2,3,5)) = \mathbb{F}_2$

3) $\widehat{HF}(\Sigma(2,3,7)) = \mathbb{F}_0^2 \oplus \mathbb{F}_{-1}$

<u>note:</u> Y is a $\mathbb{Z}HS^3 \Rightarrow \mathbb{Z}$-grading

$\gcd(p,q,r) = 1$

$$\Sigma(p,q,r) = \{ x^p + y^q + z^r \text{ in } \mathbb{C}^3 \} \cap S^5_\varepsilon(0)$$

4) $L(p,q)$

$\widehat{HF}(Y, s) = \mathbb{F}_?$ in each spin^c-str

← formula for this

5)

$$\widehat{HF}(S^1 \times S^2, s) = \begin{cases} \mathbb{F}_{1/2} \oplus \mathbb{F}_{-1/2} & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

<u>Fact:</u> if s is torsion, then $\widehat{HF}(Y,s)$ is dim ≥ 1

this implies if Y is $\mathbb{Q}HS^3$, $\dim \widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$

definition: Y a $\mathbb{Q}HS^3$ is an L-space if: $\dim \widehat{HF}(Y,\mathfrak{s}) = 1$ for all $\mathfrak{s}$

$\iff \dim \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$

definition: let $S = \bigcup_i S_i$ surface in Y

$$\chi_-(S) = \sum_i \max(-\chi(S_i), 0)$$

almost genus $2g-2$

let $\phi \in H_2(Y;\mathbb{Z})$ $(= H^1(Y;\mathbb{Z}))$

$$\chi(\phi) = \min_{[S]=\phi} \chi_-(S)$$

Thm (Ozsváth-Szabó)

$$\chi(\phi) = \min \{ \langle c_1(\mathfrak{s}) \cup P.D.(\phi), [Y] \rangle \mid \widehat{HF}(Y,\mathfrak{s}) \neq 0 \}$$

spin^c str. $\mathfrak{s}$ on Y

fundamental class

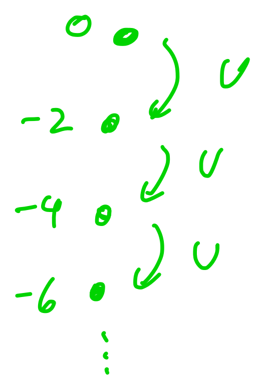
so $\widehat{HF}$ "detects" genus of ϕ

$HF^-(Y)$ this is a $\mathbb{F}[U]$-module

$$HF^-(Y) = \bigoplus_{\substack{s \text{ spin}^c \\ \text{str. on } Y}} HF^-(Y,s)$$

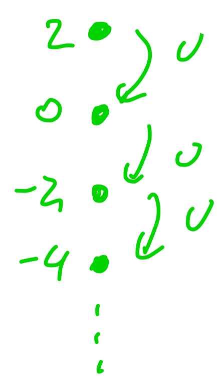
if s is torsion, then $HF^-(Y,s)$ is graded by a coset of $\mathbb{Z}$ in $\mathbb{Q}$ $(\deg U = -2)$

<u>example:</u> $HF^-(S^3)$
$\cong$
$\mathbb{F}[U]_0$

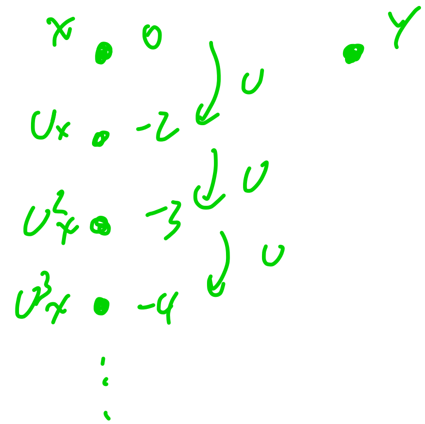


infinite tower

$HF^-(\Sigma(2,3,5))$
$\cong$
$\mathbb{F}[U]_2$



$$HF^-(\Sigma(2,3,7)) \cong \mathbb{F}[U]_0 \oplus \mathbb{F}_0$$



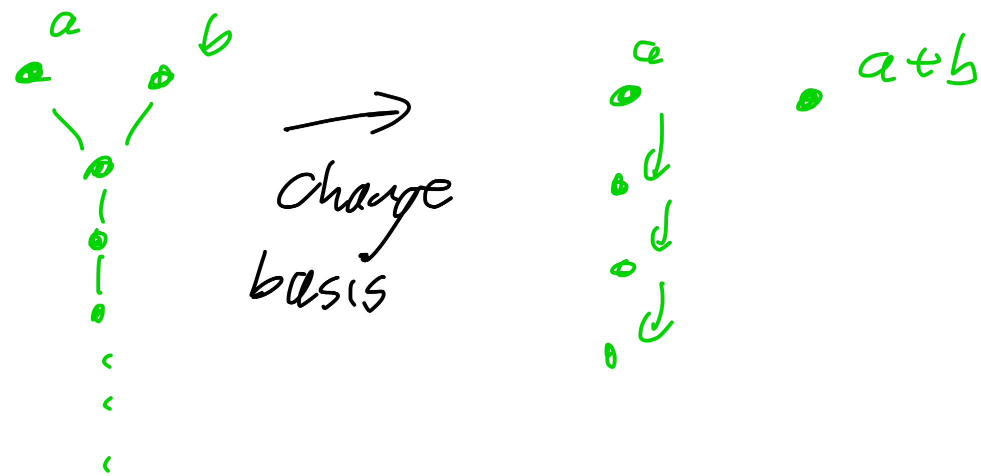
Fact: any $\mathbb{F}[U]$-modules can be written

$$\left(\bigoplus_i \mathbb{F}[U]\right) \oplus \left(\bigoplus_j \mathbb{F}[U]/U^{n_j}\right)$$

U non torsion towers

U torsion

example:



If Y is a $\mathbb{Q}HS^3$, then $HF^-(Y,s)$ has exactly <u>one</u> U-nontorsion tower

(Y L-space $\Leftrightarrow$ $HF^-(Y,s)$ has only U-nontorsion tower)

example: $HF^-(L(p,q),s) \cong \mathbb{F}[U]$ in some grading

Defⁿ: Y a $\mathbb{Q}HS^3$

$d(Y,s)$ = grading of top of "the" U-nontorsion tower

Thm (OS): Y an $\mathbb{Z}HS^3$

if Y bounds a neg. definite X then $d(Y) \geq 0$

if " " pos " " $d(Y) \leq 0$

Cobordism maps in HF

if W is a cobordism from Y_1 to Y_2

and s is a spin^c str on W

$\wedge$ or $-$ $\qquad\qquad$ $\wedge$ or $-$

$$F_{W,s} : HF(Y_1, s|_{Y_1}) \longrightarrow HF(Y_2, s|_{Y_2})$$

$$H^2(W;\mathbb{Z})$$