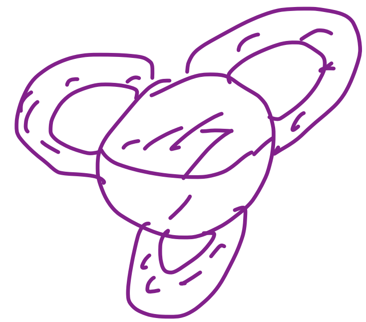
Lecture 2

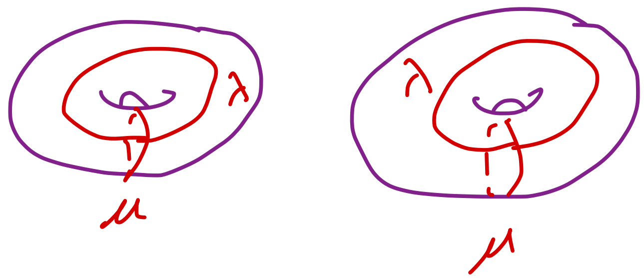
Heegaard splitting
decomposition of $Y = H_1 \cup H_2$ ← glue ∂

example: $S^3 = B^3 \cup B^3$



H_g:

example:



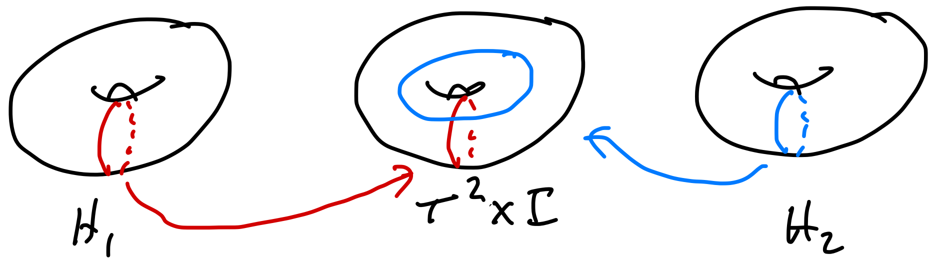
$$\left.\begin{array}{l} \mu \mapsto \mu \\ \lambda \mapsto \lambda \end{array}\right\} \longrightarrow S^1 \times S^2$$

$$\left.\begin{array}{l} \mu \mapsto \lambda \\ \lambda \mapsto \mu \end{array}\right\} \longrightarrow S^3$$

note:
$$\begin{aligned} S^3 &= \partial B^4 \\ &= \partial(B^2 \times B^2) \\ &= ((\partial B^2) \times B^2) \cup (B^2 \times \partial B^2) \\ &= (S^1 \times B^2) \cup (B^2 \times S^1) \end{aligned}$$

exercise: every 3-manifold has a Heegaard splitting another way to build M

$$H_1 \longrightarrow \underset{\Sigma_g \times \{0\} \quad \Sigma_g \times \{1\}}{\Sigma_g \times I} \longleftarrow H_2$$



only need to say where red and blue curves go

since can glue "thickened disks" along nbhd of curves

what is left is a 3-ball

only one way to do this

def^{n}: a Heegaard diagram (of genus g) is a surface Σ_g (of genus g) and two collections $\bar{\alpha}$ and $\bar{\beta}$ st.

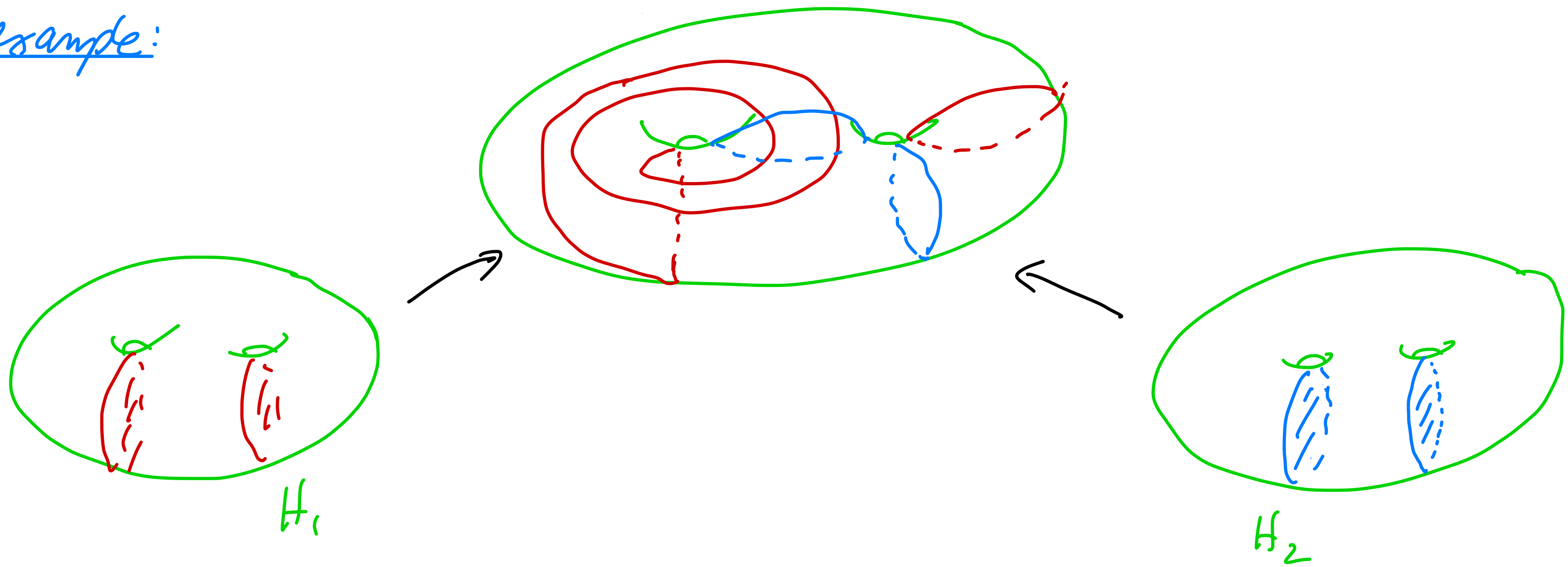
1) $\bar{\alpha}$ is a collection of g simple, closed, disjoint, non-self-intersecting curves on Σ_g

$\bar{\alpha} = (\alpha_1, \dots, \alpha_g)$ st. $\alpha_1, \dots, \alpha_g$ are lin.

independent in $H_1(\Sigma_g, \mathbb{Z})$

2) $\bar{\beta}$ " "

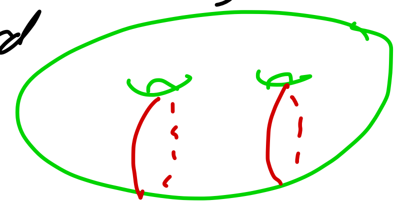
example:



exercise: attaching thickened disks (2-handles) along $\alpha_1 \dots \alpha_g$ results in boundary S^2 iff $\alpha_1, \dots \alpha_g$ are lin. indep. in $H_1(\Sigma_g)$

Remark:

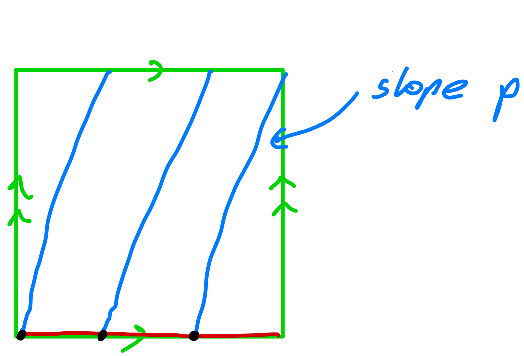
1) can always reparameterize Σ_g so one set of curves is standard



2) any 3-manifold admits many Heegaard splittings

Thm:
every 3-manifold Y admits a Heegaard splitting (diagram)
$\exists$ a set of Heegaard moves s.t. any two Heegaard splittings are related by some sequence of Heegaard moves

example



exercise This is $L(p,q)$

let Y be a 3-manifold w/ Heegaard diagram $(\Sigma_g, \bar{\alpha}, \bar{\beta})$

Consider $\mathrm{Sym}^g(\Sigma_g) = \overbrace{\Sigma_g \times \dots \times \Sigma_g}^{g \text{ copies}} / S^g$ symmetric group

set of unordered g-tuples of points on Σ_g

$\mathbb{T}_\alpha = \alpha_1 \times \dots \times \alpha_g \subset \mathrm{Sym}^g(\Sigma_g)$

set of unordered g-tuples $(x_1, \dots, x_g)$ s.t. each x_i lies on distinct α_j.

$\mathbb{T}_\beta = \beta_1 \times \dots \times \beta_g \subset \mathrm{Sym}^g(\Sigma_g)$

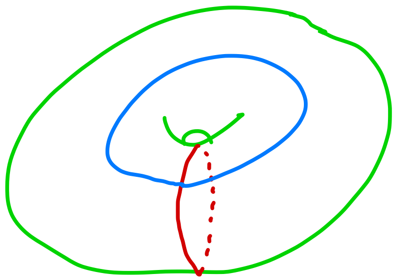
Exercise: $\mathrm{Sym}^g(\Sigma_g)$ is a $2g$-dim'l manifold

locally looks like $\mathrm{Sym}^g(\mathbb{C})$

$$\mathrm{Sym}^g(\mathbb{C}) \ni (\alpha_1, \dots, \alpha_g) \longleftrightarrow (z-\alpha_1)\dots(z-\alpha_g) = z^g + b_{g-1}z^{g-1} + \dots + b_1 z + b_0$$

look at $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$

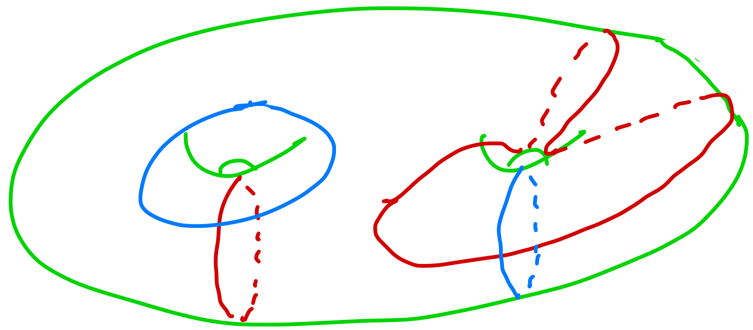
example:



genus 1 trivial

example: genus 2

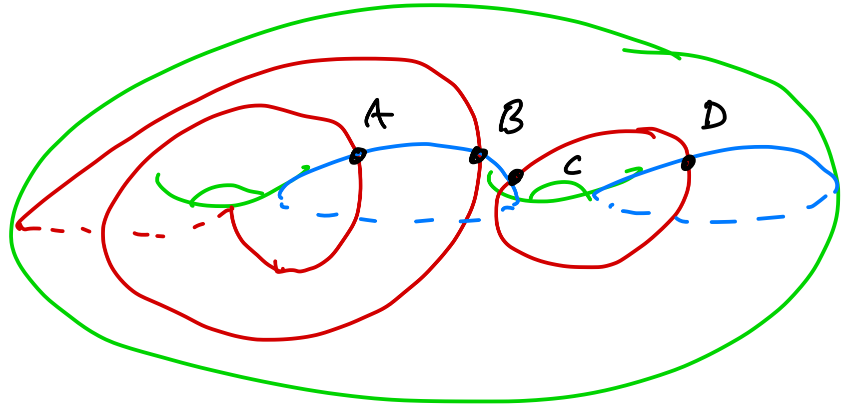
$\mathrm{Sym}^2(\Sigma_2)$



exercise: $\mathrm{Sym}^2(\Sigma_2) = T^4 \# \mathbb{C}P^2$
[Abel-Jacobi]

$\mathbb{T}_\alpha$, $\mathbb{T}_\beta$ are T^2's in here

what are $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$



the points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are g-tuples $(x_1, \dots, x_g)$

s.t. each x_i is on a distinct α_j

each x_i " " " β_k

one point must be D (since one blue only has D)

other point can't be C (since red curve already used by D)

so get (A, D) and (B, D)

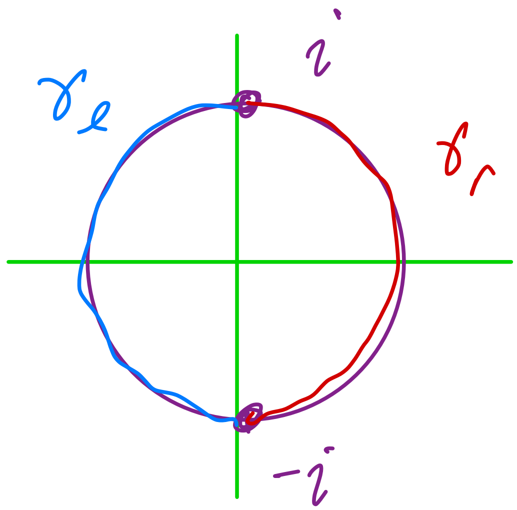
$$\widehat{CF} = \mathrm{span}_{\mathbb{F}}\{\mathbb{T}_\alpha \cap \mathbb{T}_\beta\}$$

Whitney disk

def^n: let $\bar{x}, \bar{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, where $\bar{x} = (x_1, \dots, x_g)$, $\bar{y} = (y_1, \dots, y_g)$

a Whitney disk from $\bar{x}$ to $\bar{y}$ is a continuous map

$$\phi : D^2 \to \mathrm{Sym}^g(\Sigma_g)$$

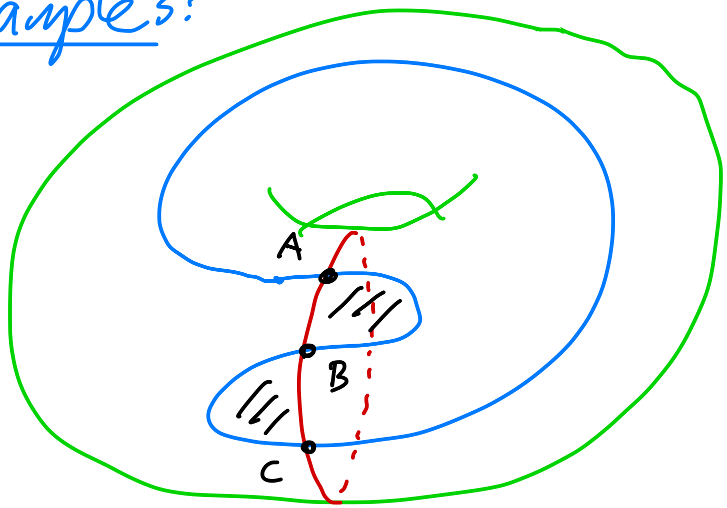


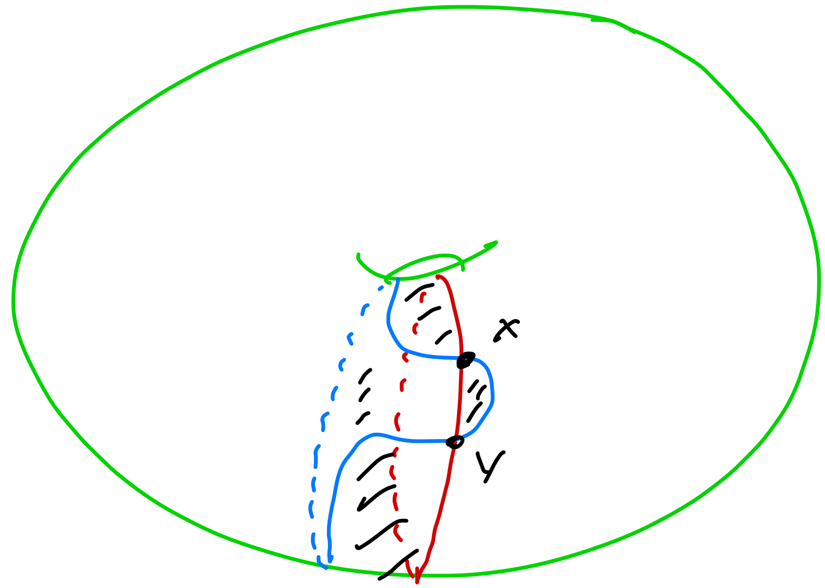
1) $\phi(-i) = \bar{x}$
$\phi(i) = \bar{y}$

2) $\phi(\gamma_r) \subset \mathbb{T}_\alpha$
$\phi(\gamma_\ell) \subset \mathbb{T}_\beta$

given $\bar{x}, \bar{y}$ denote $\pi_2(\bar{x}, \bar{y}) =$ homotopy classes of Whitney disks $\bar{x}$ to $\bar{y}$

examples:





$\pi_2(A,B) = \{1 \text{ disk}\}$

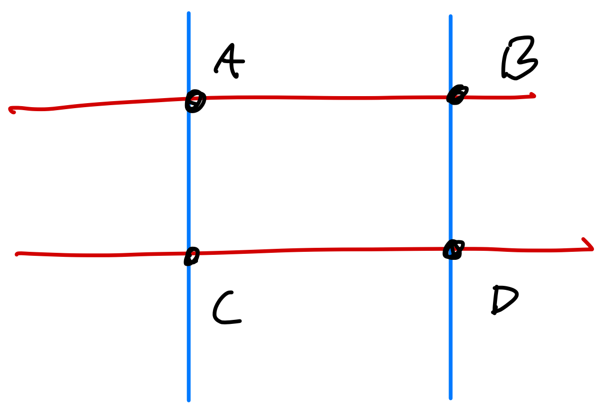
$\pi_2(C,B) = \{1 \text{ disk}\}$

π_2 others $= \emptyset$

$\pi_2(x,y) = \{2 \text{ disks}\}$

$\pi_2(y,x) = \emptyset$

examples: genus-2 portion of diagram is



generators: $\underset{\bar{x}}{\underset{\shortparallel}{(A,D)}}$, $\underset{\bar{y}}{\underset{\shortparallel}{(B,C)}}$

let r = reflection through center of square

$\mathbb{D} = \{ (x, rx) \mid x$ in above square $\} \subset \mathrm{Sym}^2(\Sigma_2)$

this contains an arc a from $\bar{x}$ to $\bar{y}$ $\subset \mathbb{T}_\alpha$

" " b from $\bar{x}$ to $\bar{y}$ $\subset \mathbb{T}_\beta$