Lecture 3 - Construction of HF

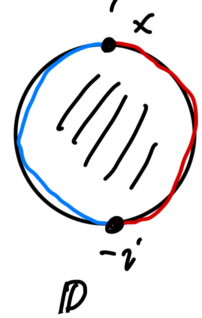
last time:

Heegaard diagram $(\Sigma_g, \bar{\alpha}, \bar{\beta})$

$\mathbb{T}_{\bar{\alpha}} \cap \mathbb{T}_{\bar{\beta}}$ in $\mathrm{Sym}^g(\Sigma_g)$

$\mathbb{T}_{\bar{\alpha}} = \alpha_1 \times \dots \times \alpha_g$, $\mathbb{T}_{\bar{\beta}} = \beta_1 \times \dots \times \beta_g$

def$^{\underline{n}}$: Whitney disk from $\bar{x}$ to $\bar{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$



$$\mathbb{D} \xrightarrow{\phi} \mathrm{Sym}^g(\Sigma_g)$$

$\pi_2(\bar{x}, \bar{y})$ = homotopy classes of disks $\bar{x}$ to $\bar{y}$

Defining $\widehat{CF}$

1) think of $\mathrm{Sym}^g(\Sigma_g)$ as a complex manifold (not really, but OK for first pass)

2) pick $z \in \Sigma_g - (\alpha_1 \cup \dots \cup \alpha_g \cup \beta_1 \cup \dots \cup \beta_g)$

base point

$V_z = \{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g) \subset \mathrm{Sym}^g(\Sigma_g)$

V_z is g-tuples with one coord $= z$

codim 2 submanifold of $\mathrm{Sym}^g(\Sigma_g)$

Def: a <u>holomorphic Whitney disk</u> is a Whitney disk which is holomorphic

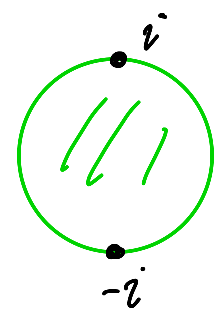
Def: for $\Phi \in \pi_2(\bar{x}, \bar{y})$ define

$$n_z(\Phi) = \text{alg intersection between } \Phi \text{ and } V_z$$

Def: let $\Phi \in \pi_2(\bar{x}, \bar{y})$

define $\mathcal{M}(\Phi) = \{\text{holomorphic Whitney disk in } \Phi\}$

$\mathcal{M}(\Phi)$ admits an $\mathbb{R}$-action



$\xrightarrow{\phi} \mathrm{Sym}^g(\Sigma_g)$

can reparameterize

this gives $\mathbb{R}$-action

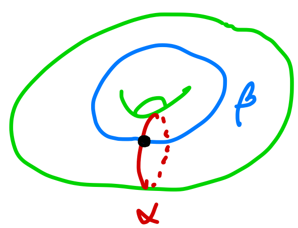
if "things" are sufficiently generic, then $\mathcal{M}(\Phi)$ is a manifold

define $\hat\partial$ "counts holomorphic disk in $Sym^g(\Sigma_g)$"

$$\hat\partial\,\bar{x} = \sum_{\substack{\bar{y}\in \mathbb{T}_\alpha\cap\mathbb{T}_\beta \\ \Phi\in\pi_2(\bar{x},\bar{y}) \\ \text{s.t. 1) } n_z(\Phi)=0 \\ \text{2) } \mathcal{M}(\Phi)\ \dim 1}} \left[\left(\#\left(\mathcal{M}(\Phi)/\mathbb{R}\right)\right) \bmod 2\right]\bar{y}$$

examples:

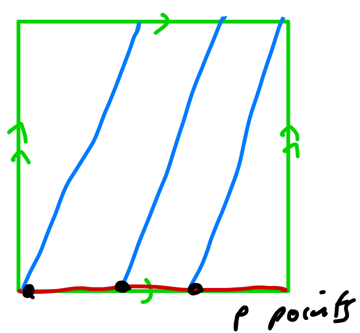
S^3



$\widehat{CF} = \mathbb{F}$ gen by one point

$\hat\partial = 0$

$L(p,1)$

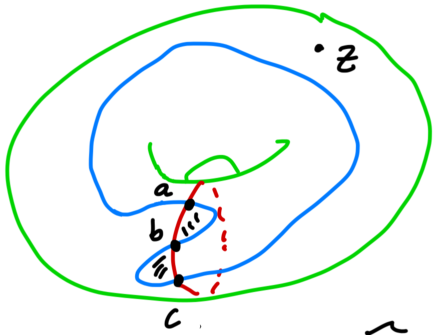


$\widehat{CF} = \mathbb{F}^p$

$\hat\partial = 0$

$\widehat{HF} = \mathbb{F}^p$

S^3



$\widehat{CF} = \text{span}\langle a, b, c\rangle$

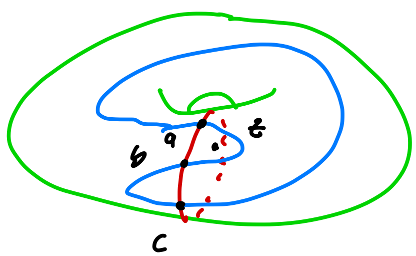
use Riemann mapping thm

each disk has one holo disk

$$\partial a = b$$
$$\partial c = b$$

$\widehat{HF} = \mathbb{F}$ gen by $a+c$

note



now $\partial a = 0$

$\partial c = b$

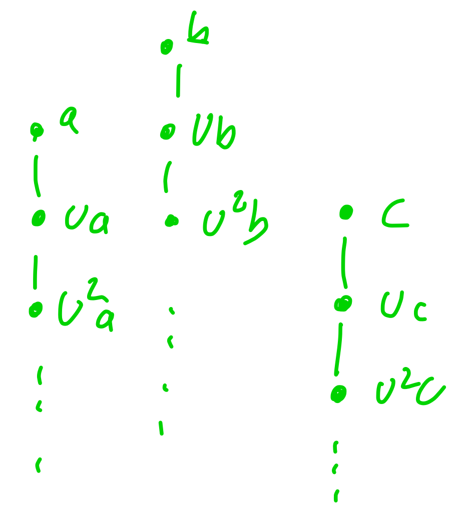
so $\widehat{HF} = \mathbb{F}$ gen by a

Thm (Ozsváth–Szabó)

1) $\hat{\partial}^2 = 0$
2) $(\widehat{CF}, \hat{\partial})$ is indep of choices

defining CF^-:

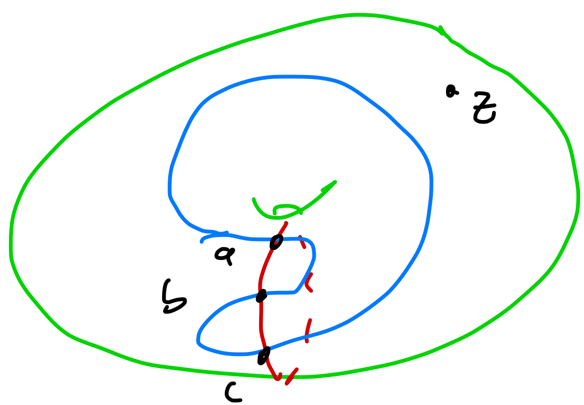
$$CF^- = \operatorname{span}_{\mathbb{F}[U]} \langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle$$



$$\partial^- \bar{x} = \sum_{\substack{\bar{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ \Phi \in \pi_2(\bar{x}, \bar{y}) \\ \dim \mathcal{M}(\Phi) = 1}} \left(\# \left(\mathcal{M}(\Phi) / \mathbb{R} \right) \right) U^{n_z(\Phi)} \bar{y}$$

extend linearly over $\mathbb{F}[U]$

example:

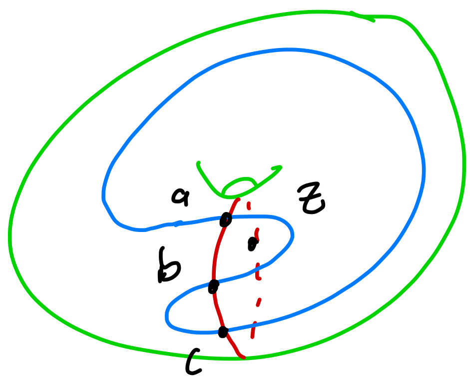


$CF^- = \mathrm{span}_{\mathbb{F}[U]}\langle a, b, c\rangle$

$\partial a = b$

$\partial c = b$

$HF^- = \mathbb{F}[U]$ gen by $a+c$



CF^- as above

$\partial a = Ub$

$\partial c = b$

cycles are b and $a + Uc$

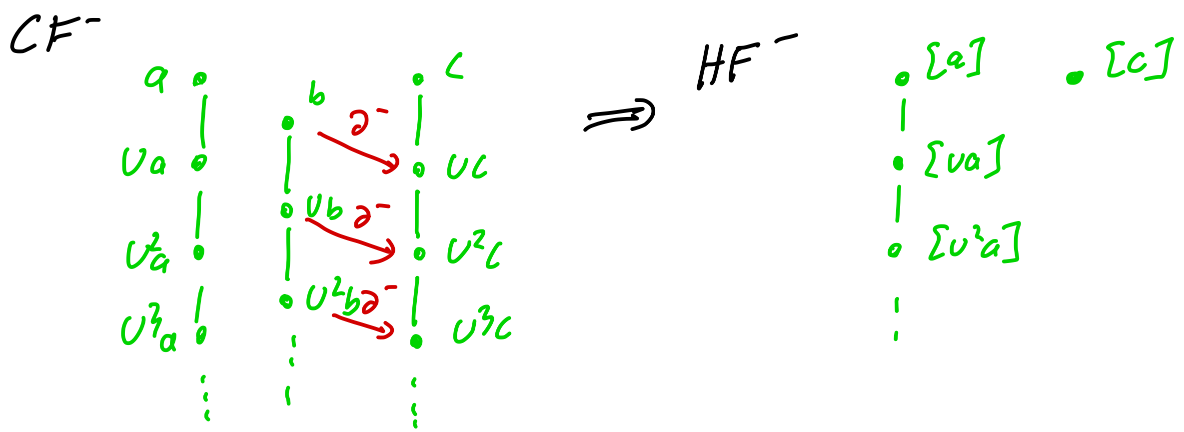
$a+Uc$ generates $HF^- = \mathbb{F}[U]$

Example: algebraic example

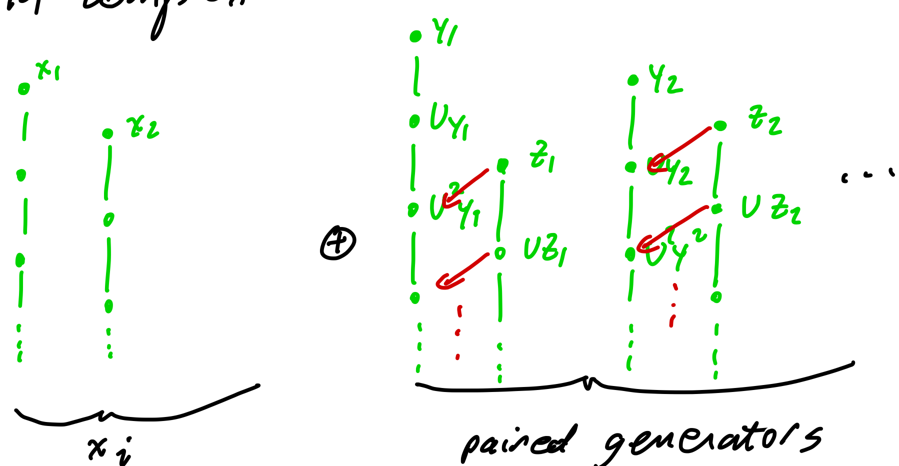
$\mathrm{span}_{\mathbb{F}[U]}\langle a, b, c\rangle$

$\partial a = 0$

$\partial b = Uc$



Th^m every free chain complex over $\mathbb{F}[U]$ is homotopy equivalent to a chain complex



note: $(\widehat{CF}, \hat{\partial})$ is just (CF^-, ∂^-) with $U=0$

so each free tower gives an $\mathbb{F}$ in $\widehat{HF}$
each torsion tower gives 2 $\mathbb{F}$s in $\widehat{HF}$

if you don't use base point then for homology spheres always get $\mathbb{F}$

Idea for grading: $\bar{x}, \bar{y}$

$\Phi \in \pi_2(\bar{x}, \bar{y})$

compute $\dim \mathcal{M}(\Phi)$

define "relative" grading to be $\dim \mathcal{M}(\Phi)$

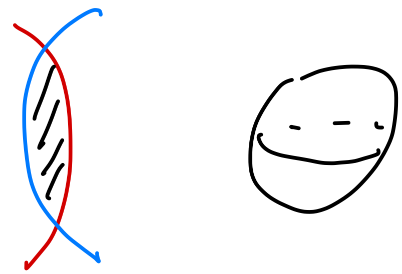
if $g \geq 2$ there are many elements of $\pi_2(\bar{x}, \bar{y})$
and these have different $\mathcal{M}(\Phi)$ dimensions

Fact:

$$H_2(\mathrm{Sym}^g(\Sigma_g)) = \pi_2(\mathrm{Sym}^g(\Sigma_g)) = \mathbb{Z}$$

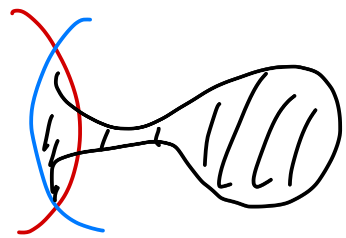
in $\mathrm{Sym}^g(\Sigma_g)$

given



gen $\pi_2(\mathrm{Sym}^g(\Sigma_g))$

also get



expected dim $(\Phi \# S^2)$ = expected dim $(\Phi) + 2$

(by reparam. of S^2)

so how do define "relative" grading

but when you sum with S^2 increase $\cap$ with V_z

so this fixed "rel" grading problem