Lecture 4

Introduction to knot Floer

(applications, examples, formal properties)

aside involving spin^c structures

$\Phi \in \pi_2(\bar{x}, \bar{y})$

$\bar{x}, \bar{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

there may be a homological restriction to the existence of Whitney disk from $\bar{x}$ to $\bar{y}$

pick arc a with $\partial a = \bar{x} - \bar{y}$ in $\mathbb{T}_\alpha$

" " b with $\partial b = \bar{x} - \bar{y}$ in $\mathbb{T}_\beta$

if $\underbrace{a-b}_{} \in \underbrace{H_1(\mathrm{Sym}^g(\Sigma_g)) / \iota_*(H_1(\mathbb{T}_\alpha)) \oplus \iota_*(H_1(\mathbb{T}_\beta))}_{}$

independent of choice in

exercise: $= H_1(\Sigma_g) / \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \rangle$

$= H_1(Y)$

so $\widehat{CF}$ CF^- split over $H_1(Y)$ "spin^c structures"

Question: about knots

1) Distinguishing knots

   knot Floer strictly stronger that Alexander poly.

2) Knot detection:

   unknot, trefoil (Right and Left),
   figure 8, 5_1, 5_2, misc. others + some links

3) Detects Seifert genus

4) Detects fiberdness

Questions: "things between" knots

1) sliceness/slice genus/coordance

   knot Floer bounds slice genus from below

   $\mathcal{C}$ concordance group

   - How big is $\mathcal{C}$?
   - What kind of torsion is in $\mathcal{C}$?
   - Generating questions of types of knots

2) Exotic phenomena for slice surfaces

Σ, Σ' slice surfaces for K s.t.

Σ, Σ' topologically isotopic

Σ, Σ' smoothly <u>not</u> isotopic

$\widehat{HFK}$: bigraded vector space $\mathbb{F}$

$\mathbb{Z} \oplus \mathbb{Z}$

Alexander grading

Maslov/Homological grading

$$\widehat{HFK}(K) = \bigoplus_{s \text{ Alexander grading}} \underbrace{\widehat{HFK}(K,s)}_{= \bigoplus_i \widehat{HFK}_i(K,s)}$$

<u>example:</u>

unknot $\quad s=0 \quad \mathbb{F}_0$

trefoil (right handed) $\quad s=1 \quad \mathbb{F}_2$

$s=0 \quad \mathbb{F}_1$

$s=-1 \quad \mathbb{F}_0$

figure 8

$s=-1 \quad \mathbb{F}_1$

$s=0 \quad \mathbb{F}_0^3$

$s=-1 \quad \mathbb{F}_{-1}$

$Wh_+(\text{trefoil})$

$s=1 \quad \mathbb{F}_{-1}^2 \oplus \mathbb{F}_0^2$

$s=0 \quad \mathbb{F}_{-2}^4 \oplus \mathbb{F}_{-1}^3$

$s=-1 \quad \mathbb{F}_{-3}^2 \oplus \mathbb{F}_{-2}^2$

Thm:

$$\sum_{s\in\mathbb{Z}} \chi(\widehat{HFK}(K,s))\,t^s = \Delta_K(t)$$

Alexander polynomial

example:

unknot $\quad \Delta = 1$

trefoil $\quad \Delta = t - 1 + t^{-1}$

figure 8 $\quad \Delta = -t + 3 - t^{-1}$

any untwisted Whitehead double has $\Delta = 1$

Thm:

$g_3(K) \geq \text{degree}(\Delta_K)$

$g_3(K) = \max_{s \in \mathbb{Z}} \{ s \mid \widehat{HFK}_s(K,s) \neq 0 \}$

Thm:

If K is fibered then $\Delta_K(t)$ is monic

K is fibered $\iff \widehat{HFK}(K, g_3(K))$ has dim 1

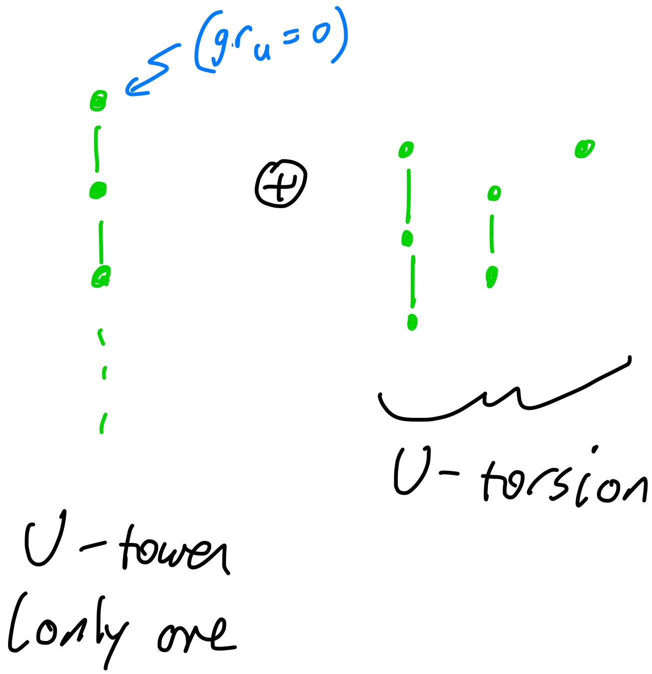
$HFK^-(K)$: bigraded $\mathbb{F}[U]$-module

$HFK^-(K) = \bigoplus_s HFK^-(K,s)$

multiplication by U decreases hom. grading by 2 and Alex. grading by 1

define gr_u = homological grading

gr_v = homological grading -2 Alexander grading so mult by U preserves gr_v grading

$HFK^- =$ 

$\tau(K) = -\frac{1}{2}(gr_v\text{-grading in which } \mathbb{F}[U]\text{-tower is located})$

Thm:

K_1 concordant to $K_2 \Rightarrow \tau(K_1) = \tau(K_2)$

$\tau : \mathcal{C} \to \mathbb{Z}$ homomorphism

example: $\tau(\text{unknot}) = 0$

$\tau(\text{trefoil}) = 1$

$\tau(Wh_+(\text{trefoil})) = 1$

Open problem: is $Wh_-(\text{right trefoil})$ slice

Th^m:

any knot with Alexander poly 1 is topologically slice

so $Wh_+(RHT)$ topologically, not smoothly, slice

Def: $Ord_U(K) = \min_n \{ \text{s.t. } U^n HFK^-_{red}(K) = 0 \}$

i.e. max length of U-torsion tower

Th^m (JMZ):

If K is ribbon knot, then

\# of bands in any ribbon disk $\geq Ord_U(K)$

Exotic pairs of disks/surfaces

Want Σ, Σ' for K s.t.

1. Σ and Σ' are topologically isotopic rel K

ex: if Σ, Σ' disks with

$$\pi_1(B^4 - \Sigma) = \pi_1(B^4 - \Sigma') = \mathbb{Z}$$

then Σ, Σ' topologically isotopic

② if Σ is a knot cobordism from K_1 to K_2 then get

$$F_\Sigma : HFK(K_1) \to HFK(K_2)$$

if Σ, Σ' are smoothly isotopic rel boundary $F_\Sigma = F_{\Sigma'}$