The L-space Conjecture

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I Conjecture

M closed (conn. oriented, irreducible) 3-mfd

L-space Conjecture

M is LO $\Leftrightarrow$ M is CTF $\Leftrightarrow$ M is NLS

(Boyer - G - Watson; Juhász)

LO: a group $G \neq 1$ is left-orderable if $\exists$ linear order $<$ on G st $g < h \Rightarrow fg < fh$

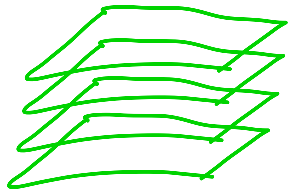
M is LO if $\pi_1(M)$ is left orderable

$\Leftrightarrow$

$\exists$ non-trivial homeo $\pi_1(M) \to \mathrm{Homeo}_+(\mathbb{R})$

(Boyer - Rolfsen - Wiest)

CTF: M is CTF if it has a co-orientable taut foliation $\mathcal{F}$



coordinate chart $\mathbb{R}^2 \times \mathbb{R}$

$M = \coprod$ leaves of $\mathcal{F}$

taut: $\exists$ simple closed curve $\gamma \subset M$ that meets each leaf of $\mathcal{F}$ transversely

NLS: Heegaard Floer Homology (Ozsváth–Szabó)

$H_1(M)$ finite, M $\mathbb{Q}HS^3$

$\widehat{HF}(M)$: finite dim'l vector space $/ \mathbb{F}_2$

$$\widehat{HF}(M) = \widehat{HF}_+(M) \oplus \widehat{HF}_-(M)$$

$$\left|\dim \widehat{HF}_+(M) - \dim \widehat{HF}_-(M)\right| = |H_1(M)|$$

$$\therefore \dim \widehat{HF}(M) \geq |H_1(M)|$$

M is an L-space if $=$

NLS is not a L-space

Remarks:

(1) $H_1(M)$ is finite $\Rightarrow$ M is $\begin{cases} \text{LO (BRW)} \\ \text{CTF (Gabai)} \\ \text{NLS (Def}^{\text{n}}\text{)} \end{cases}$

(2)

$$\begin{array}{ccc} & M \text{ is CTF} & \\ & \Downarrow \;\; \not\Uparrow & \\ M \text{ is LO} & \underset{\not\Leftarrow}{\Rightarrow} & \pi_1(M) \text{ is finite} \\ & \Uparrow \;\; \not\Downarrow & \\ & M \text{ is NLS} & \end{array}$$

II Th$^{\text{m}}$ (OS, Kazez-Roberts, Bowden)

CTF $\Rightarrow$ NLS

(proof goes through contact structures)

CTF $\Rightarrow$ LO ?

$\mathcal{F}$ c.t.f. on M

$\Rightarrow$ $\tilde{\mathcal{F}}$ on $\tilde{M} \cong \mathbb{R}^3$

with leaves $\cong \mathbb{R}^2$

$\pi_1(M)$ acts on $(\tilde{M}, \tilde{\mathcal{F}})$

(A) $\tilde{M}/(\text{leaves of } \tilde{\mathcal{F}} \sim pt) = \Lambda$ leaf space

Λ is a simply connected 1-manifold

$\pi_1(M)$ acts on Λ

if $\Lambda \cong \mathbb{R}$ then $\pi_1(M) \to \mathrm{Homeo}_+(\mathbb{R})$
non-trivial

$\therefore$ M is LO

But: Λ not nec. Hausdorff

(B) space M hyperbolic: every leaf λ of $\tilde{\mathcal{F}} \sim \mathbb{H}^2 \to$ circle at ∞, S^1_λ

$\mapsto$ action of $\pi_1(M)$ on <u>universal</u>

<u>circle</u> S^1_{univ}

(Thurston; Calegari-Dunfield)

$$\begin{tikzcd}
 & & \mathrm{Homeo}_+(\mathbb{R}) & & \\
 & & \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \arrow[u, phantom, "\subseteq" sloped] \arrow[r] & \mathrm{Homeo}_+(S^1) \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Z} \arrow[ur] & & & \\
 & & \pi_1(M) \arrow[uu, dashed, "\tilde\rho"] \arrow[uur, "\rho"'] & &
\end{tikzcd}$$

$\tilde\rho$ exists iff $e(\rho) \in H^2(\pi_1(M))$
$\cong H^2(M) \cong H_1(M)$
is zero

Euler class

So <u>Thm</u> M hyperbolic $\mathbb{Z}HS^3$ (i.e. $H_1(M)=0$)
then M is CTF $\Rightarrow$ M is infinite

<u>Conj (OS):</u> let M be a $\mathbb{Z}HS^3$

Then M is NLS $\Leftrightarrow \pi_1(M)$ is finite

($i.e.$ $M \not\cong S^3$ or PHS)

LO $\Rightarrow$ CTF ?

Thm (Li) if $g(M) = 2$ then M LO $\Rightarrow$ CTF

(S. Rasmussen)

LO $\Rightarrow$ NLS ? $\quad \left|\dim \widehat{CF}_+ - \dim \widehat{CF}_-\right| = |H_1(M)|$

$\therefore \dim \widehat{CF}(M) \geq |H_1(M)|$

M a strong L-space if $=$

Thm (Levine-Lewallen): M LO $\Rightarrow$ M is not a strong L-space

NLS $\Rightarrow$ LO/CTF ???

III Geometrization Conjecture $\Rightarrow$ M is either
Seifert Fibered
toroidal ($\supset$ incomp. torus)
hyperbolic

<u>Th^{m} (BRW; Lisca–Stipsicz, BGW)</u>

the L-space conjecture is true for Seifert manifolds

<u>M toroidal</u>: Th^{m}: the L-space conjecture true for graph mfds (all parts of JSJ seif fib.)

Boyer–Clay, Hanselmann–J. & S. Rasmussen–Watson
S. Rasmussen

M a $\mathbb{Q}HS^3$ $\quad$ $T \subset M$ incompressible torus

$M = X_1 \cup_T X_2 \quad X_i$: $\mathbb{Q}$HSolid Torus $i=1,2$

X $\mathbb{Q}HST$, α a slope $\subset \partial X$

$\exists$ notion of α <u>being $*$ detected in X</u>

($* =$ LO, CTF, NLS)

$\mathcal{D}_*(X) = \{\alpha \in \partial X : \alpha$ $*$-detected in $X\}$

$* =$ LO [BC]

NLS [HRW]

CTF [Boyer-G-Hu]

<u>Th^m</u>: $M = X_1 \cup_T X_2$, if $\mathcal{D}_*(X_1) \cap \mathcal{D}_*(X_2) \neq \emptyset$

then M is $*$

($* =$ NLS <u>converse</u> is true)

<u>Th^m</u>: M toroidal $\mathbb{Z}HS^3$, then M is
- NLS Eftekhari [HRW]
- LO [BGH]
- CTF if some X_i is fibered

<u>M hyperbolic</u>: $\pi_1(M) \longrightarrow PSL(2, \mathbb{C})$

<u>approach</u>: conjugate to rep^{n} to $PSL(2,\mathbb{R})$

$\longmapsto \widetilde{PSL}(2,\mathbb{R}) \dots \Rightarrow M$ is LO.

(Dunfield)

IV $\exists$ many results on 3-mfd that are

(1) Dehn surgery on knots : $K(r)$ $r \in \mathbb{Q}$

(2) n-fold cyclic branched covers of K : $\Sigma_n(K)$

$K(r)$ is NLS $\Leftrightarrow$ no r or $r \in [2g-1, \infty)$

few things known abt (2)

eg. (1) Th^{m}: K composite, then $\forall r \in \mathbb{Q}$ $K(r)$ is

- NLS (Krcatovich)
- LO (BGH)
- CTF if K has a fibered summand (Delman-Roberts)

(2) Thm: K prime satellite knot

Then $\forall n \geq 2$

$\Sigma_2(K)$ is LO & NLS (BGH)

CTF if companion is fibered

$\Sigma_n(5_2)$ $n = 6, 7, 8$

known to be NLS

is it LO? is it CTF?