# Immersed curve bordered invariants from knot Floer homology

Tech Topology Summer School, July 2023

Jonathan Hanselman

Princeton University

**Plan:** I) Knot Floer as immersed curves

- curve invariant
- surgery formula
- invariance
- pairing theorem

II) Application: L-space criterion for toroidal gluing

**Remarks:**

- some done, some in progress
- built on earlier joint work w/ 1. J. Rasmussen 2. Watson (application from that work)

**Advantages of new approach:**

- not restricted to hat version
- not restricted to $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

} not helpful for L-space conjecture

☆ • don't need to define bordered Floer homology

Recall:

## Heegaard Floer homology

often field, could be $\mathbb{Z}$

closed or. 3-mfd $Y$ $\rightsquigarrow$ graded module $HF^-(Y)$ over $\mathbb{F}[U]$

uses pointed Heegaard diagram

$$= \bigoplus_{s \in Spin^c(Y)} HF^-(Y;s)$$

$$\boxed{\text{simpler version: set } U=0 \rightsquigarrow \widehat{HF}(Y)}$$

# Knot Floer homology

uses doubly pointed Heegaard diagram

bigraded chain cpx over $\mathbb{F}[U,V] =: R^-$

$$CFK_{R^-}(Y,K) = \bigoplus_{s \in Spin^c(Y)} CFK^-(Y;s)$$

$K \subset Y$

seq. of Heegaard moves

Flip isomorphism

both $\cong$ to $HF^-(Y)$

$$\Psi : H_*(\underbrace{U=1}_{\text{ignore } w \text{ basepoint}} \text{ complex}) \longrightarrow H_*(\underbrace{V=1}_{\text{ignore } z \text{ basepoint}} \text{ complex})$$

$$\Psi = \bigoplus \Psi_s : CFK^-(Y,K;s)\big|_{U=1} \longrightarrow CFK^-(Y,K;s+PD(K))\big|_{V=1}$$

<u>Note</u>: $\Psi$ boring when $Y = S^3$

$$\boxed{\text{simpler version: set } UV=0 \rightsquigarrow CFK_{\hat{R}}(Y,K)}$$

$\left(\hat{R} = \mathbb{F}[U,V]/UV=0\right)$

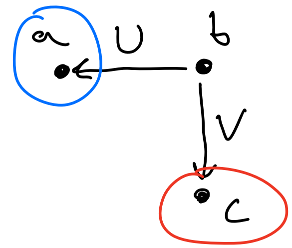
# Knot Floer homology

## Examples:

$\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

- $T_{2,3} \subset S^3 \rightsquigarrow$

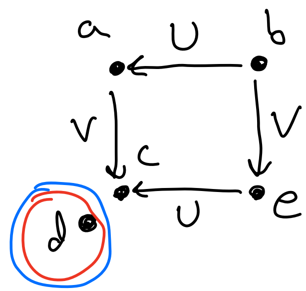
$$\partial(a) = 0$$
$$\partial(b) = Ua + Vb$$
$$\partial(c) = 0$$




$$\Psi : \langle c \rangle \xrightarrow{\cong} \langle a \rangle$$

- $\text{Fig } 8 \subset S^3 \rightsquigarrow$

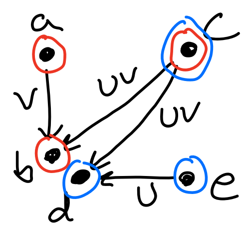
$$\partial(a) = Vc$$
$$\partial(b) = Ua + Ve$$
$$\partial(c) = 0$$
$$\partial(d) = 0$$
$$\partial(e) = Uc$$




$$\Psi : \langle d \rangle \xrightarrow{\cong} \langle d \rangle$$

- dual knot in $S^3_1(-T_{2,3}) \rightsquigarrow$

$$\partial(a) = Vb$$
$$\partial(b) = 0$$
$$\partial(c) = UVb + UVd$$
$$\partial(d) = 0$$
$$\partial(e) = Ud$$



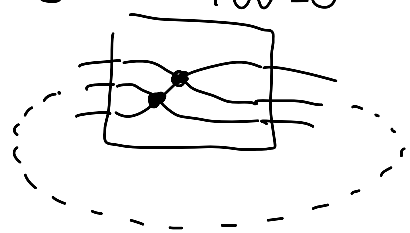

$$\Psi : \mathrm{Span}(\{a, b, c\}) \longrightarrow \mathrm{Span}(\{c, d, e\})$$
$$a \longmapsto c$$
$$b \longmapsto d$$
$$c \longmapsto e$$

Theorem 1 (H.) Given $K \subset Y^3$

(a) $\begin{pmatrix} \text{cpx over } \mathbb{F}[U,V] \\ + \\ \text{flip isomorphism} \end{pmatrix} \longleftrightarrow$ decorated immersed multicurve $(\Gamma, b)$ in marked torus $T$

- grading
- bounding chain $b$ (lin. comb. over $\mathbb{F}[UV]$ of self int. pts of $\Gamma$)

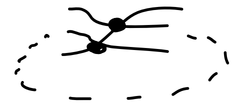
(b) If $\mathbb{F}$ is field,
we can choose $(\Gamma, b)$ so that $\hat{b} = b|_{UV=0}$ contains only points of following form:



such $(\Gamma, \hat{b})$ is well defined (up to homotopy of $\Gamma$)
for each chain homotopy equiv. class of complex / flip map

**Theorem** (H.) Given $K \subset Y^3$

(a) [Knot Floer data] $\longleftrightarrow$ decorated immersed multicurve $(\Gamma, b)$ in marked torus $T$

(b) $\mathbb{F}$ field $\Rightarrow \exists\ (\Gamma, b)$ so $\hat{b}$ contains only 

such $(\Gamma, \hat{b})$ is unique

**Rmks:**

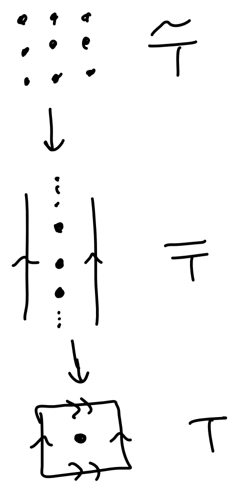
- (a) $\longleftarrow$ is immediate (Floer homology)
- (a) $\longrightarrow$ is easy (but messy representative)

So theorem is really (b)

- $UV=0$ knot Floer data $\longrightarrow (\Gamma, \hat{b})$ invariant of $K$
- $UV=0$ knot floer data "$\longleftrightarrow$" bordered Floer homology of $Y \setminus K$

  expected

- $UV=0$, $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$ version of Thm 1 is analogous to (equivalent to) Theorem of H.-Rasmussen-Watson




- Convenient to work in covering space of $T$

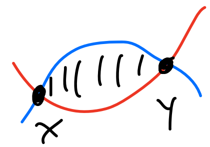
# Aside: Floer homology of (decorated) curves in surfaces

- Given curves $\Gamma_0 \pitchfork \Gamma_1$ in surface $\Sigma$, define chain cpx

  $CF(\Gamma_0, \Gamma_1)$:
  - generated by $\Gamma_0 \cap \Gamma_1$

    coefficients $\mathbb{F}$, $\mathbb{F}[U]$, or $\mathbb{F}[U,V]$ based on # of marked points

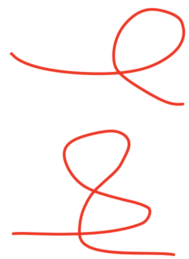
  - differential: bigons



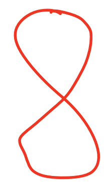

$\Rightarrow \partial x = c\, y$

- $\Gamma_0, \Gamma_1$ may be immersed as long as they satisfy

  **zero monogon condition**:



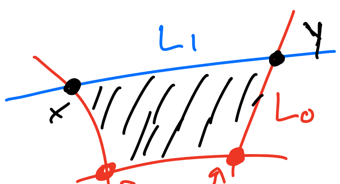
bad



good

- For $(\Gamma_0, b_0), (\Gamma_1, b_1)$

  differential counts "generalized bigons"

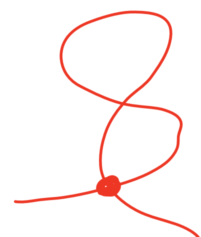



$\in b_0$

orientation agrees before/after

$(\Gamma_i, b_i)$ must satisfy

zero generalized monogon condition

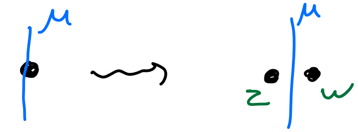


good

# Decorated curves $\longrightarrow$ cpx over $\mathbb{F}[U,V]$ + flip map

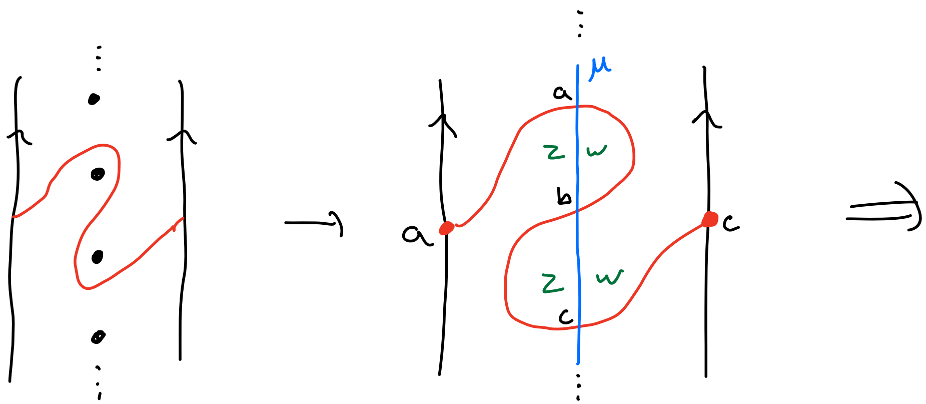
in $\mathcal{T}$ = infinite marked cylinder

1. Draw vertical line $\mu$ through marked points
2. Replace each marked point with pair




3. complex: Floer homology of $(\Gamma, b)$ with $\mu$
4. flip map: (roughly) how curve matches up on back of cylinder

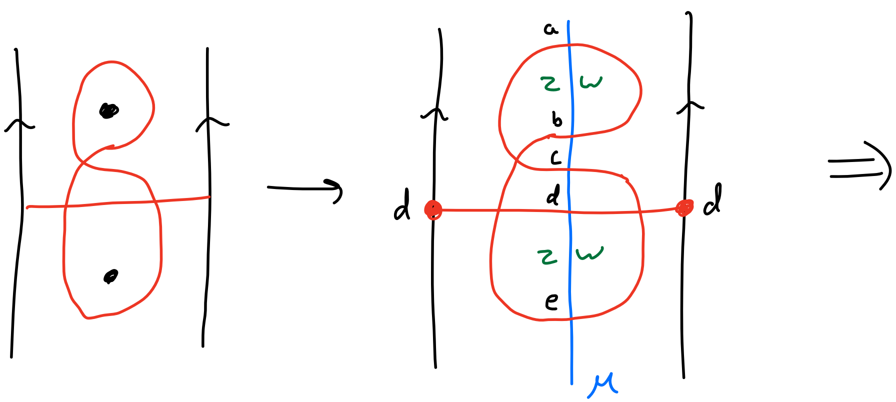
e.g.




$$\begin{tikzcd} a & b \arrow[l, "U"'] \arrow[d, "V"] \\ & c \end{tikzcd}$$

$\psi : c \longmapsto a$

e.g.

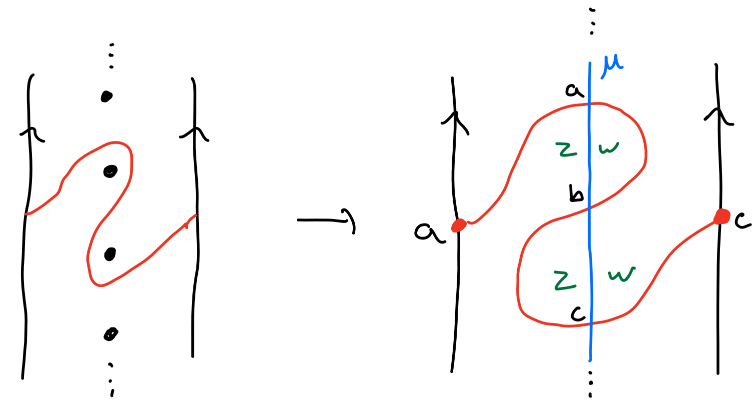



$$\begin{tikzcd} a \arrow[d] & b \arrow[l] \arrow[d] \\ c & e \arrow[l] \end{tikzcd} \quad d$$

$\psi : d \longmapsto d$

# Decorated curves $\longrightarrow$ cpx over $\mathbb{F}[U,V]$ + flip map

e.g.

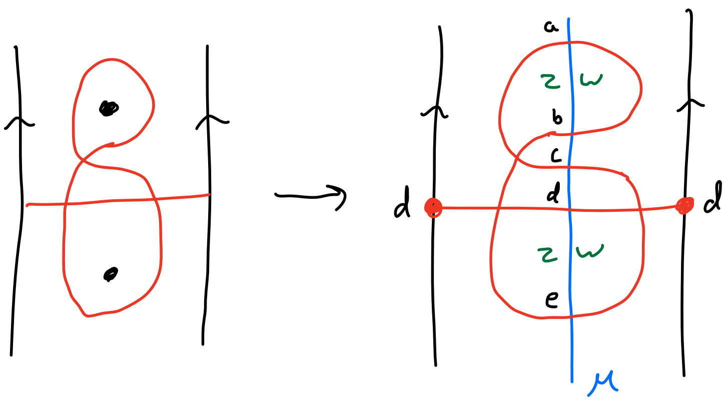



$\Longrightarrow$

$$\begin{tikzcd} a & b \arrow[l, "U"'] \arrow[d, "V"] \\ & c \end{tikzcd}$$

$\psi: c \longmapsto a$

e.g.



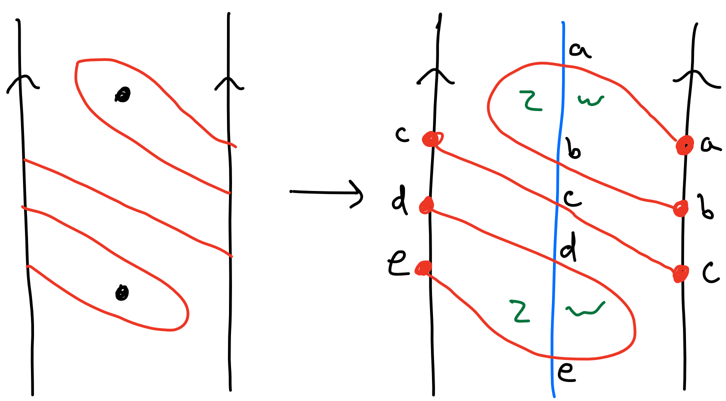

$\Longrightarrow$

$$\begin{tikzcd} a \arrow[d] & b \arrow[l] \arrow[d] \\ c & e \arrow[l] \end{tikzcd}$$

$d \bullet$

$\psi: d \longmapsto d$

e.g.




$\Longrightarrow$

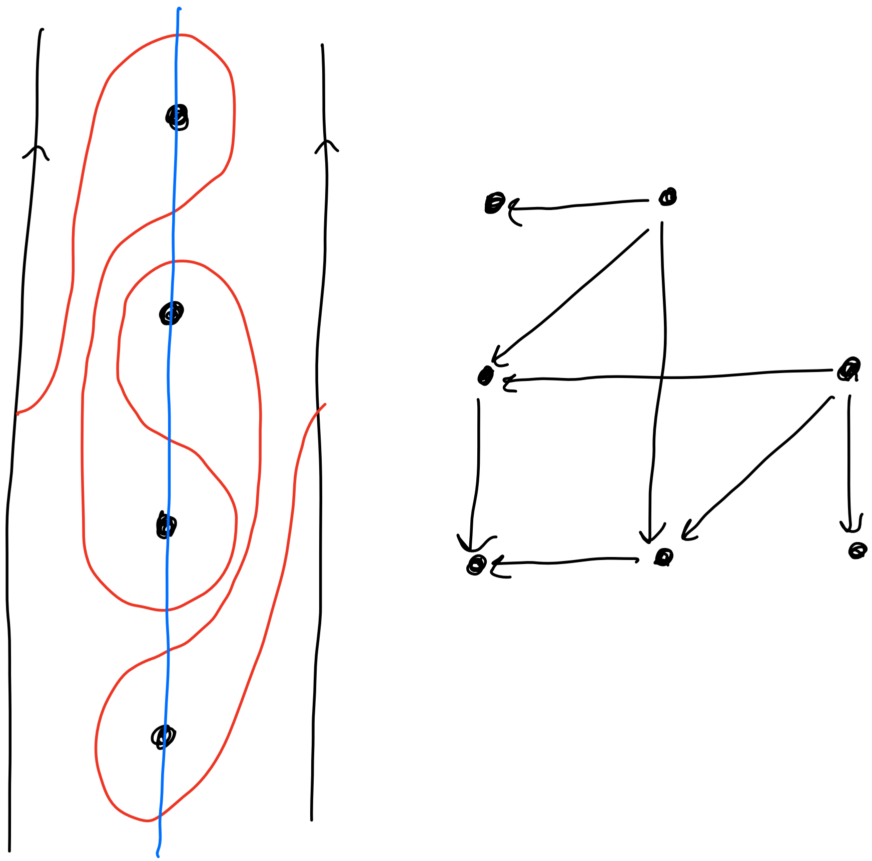
$$\begin{tikzcd} a \arrow[d, "V"'] & & c \arrow[dll, "UV"'] \arrow[dl, "UV"] \\ b & d & e \arrow[l, "U"] \end{tikzcd}$$

$\psi: \begin{array}{l} a \longmapsto c \\ b \longmapsto d \\ c \longmapsto e \end{array}$

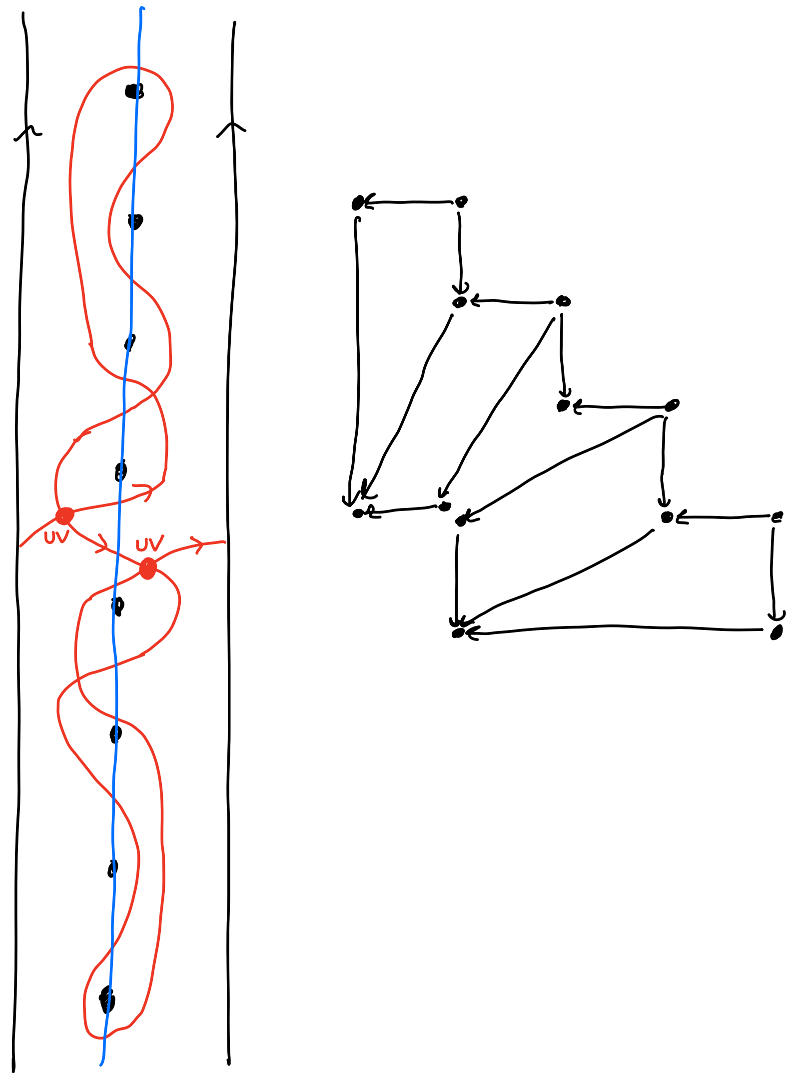
# More examples

(component of inv't for)

$T_{2,3;2,1}$



$T_{2,9} \# T_{2,3;2,5}$



# Surgery formula

Ozsváth-Szabó: $\exists$ way to compute $HF^-(Y^3_{p/q}(K))$ from knot Floer data for $K \subset Y$

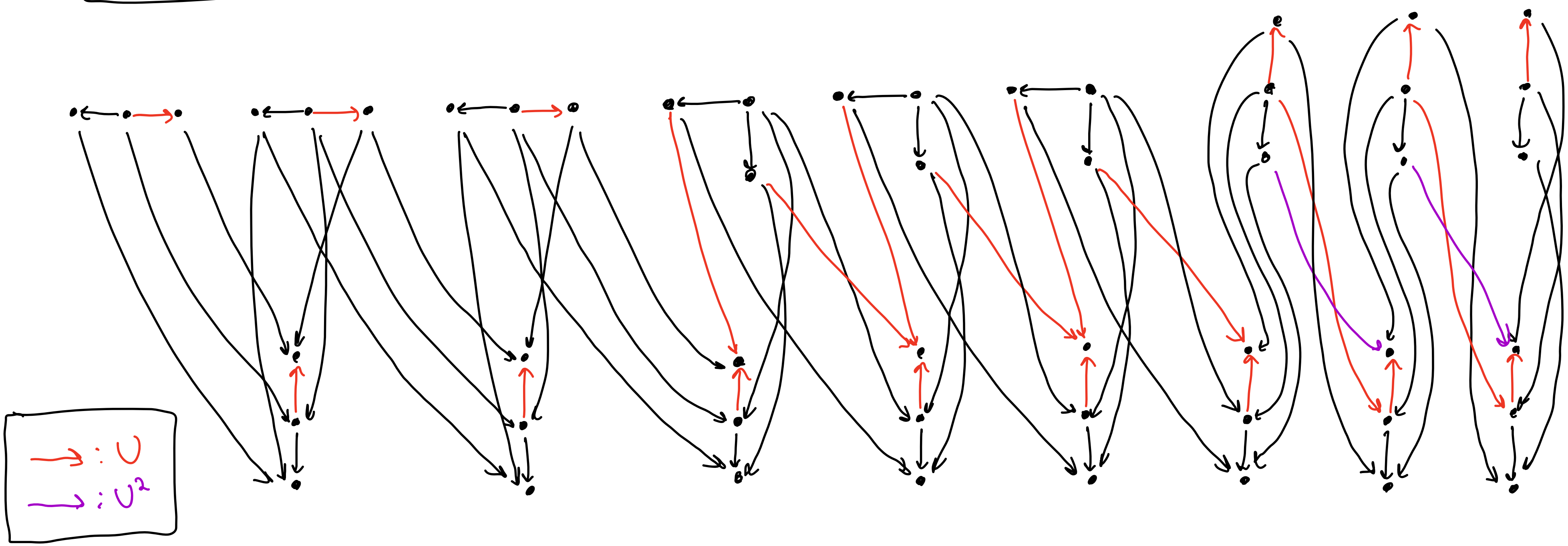
"Mapping cone formula"

$$HF^-(Y^3_{p/q}(K)) = H_*\left( \begin{array}{c} \bigoplus_{n\in\mathbb{Z}} A^-_{\lfloor \frac{n}{q} \rfloor} \\ v \downarrow \quad \downarrow h \\ \bigoplus_{n\in\mathbb{Z}} B_{\lfloor \frac{n}{q} \rfloor} \end{array} \right)$$

subcpxs of $CFK^-$

$\leftarrow$ uses $\Psi$

$\boxed{\text{For } S^3_{1/3}(T_{2,3}):}$



# Surgery formula

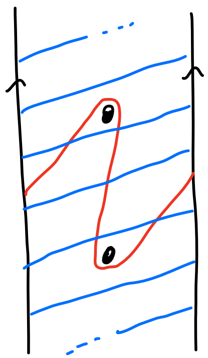
**Theorem** (H.): $HF^-(Y^3_{p/q}(K))$ is isomorphic to the Floer homology (in marked torus $T$) of $(\Gamma, b)$ with $\ell_{p/q}$.

knot Floer data for $K$

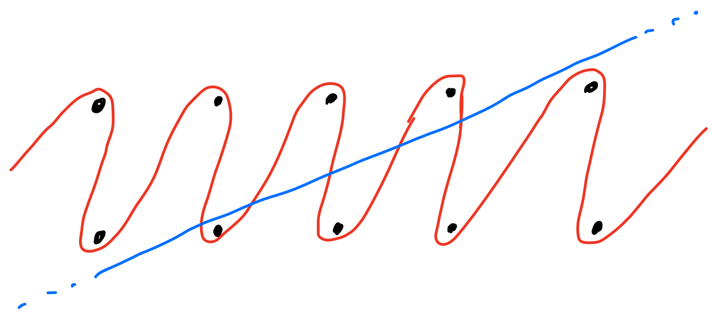
s.c.c. of slope $p/q$

For $S^3_{1/3}(T_{2,3})$:

$T$:



$\tilde{T}$:



dim $\widehat{HF} = 5$

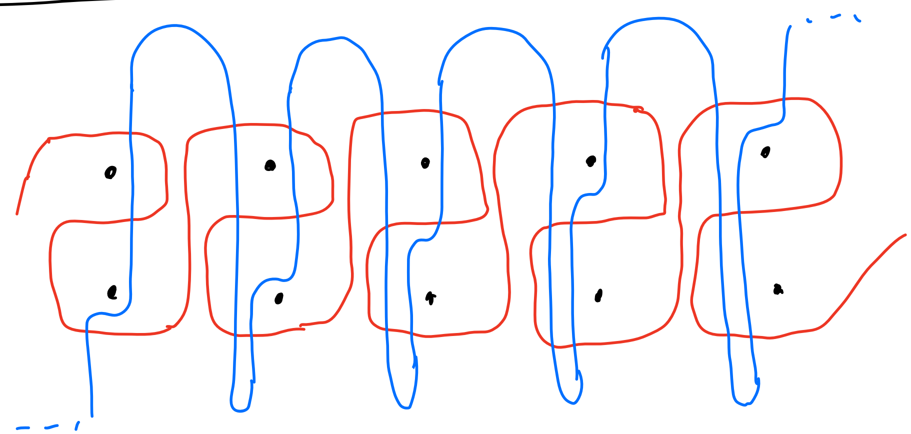
(minimal intersection number)

$$HF^- \cong H_*\left(\begin{tikzcd}[column sep=small] \bullet \arrow[dr, "U"'] & & \bullet \arrow[dl, "U"] \arrow[dr, "U"'] & & \bullet \arrow[dl, "U"] \\ & \bullet & & \bullet & \end{tikzcd}\right) \cong \mathbb{F}[U] \oplus \mathbb{F}^2$$

pf idea:

perturb so

Floer cpx = mapping cone cpx

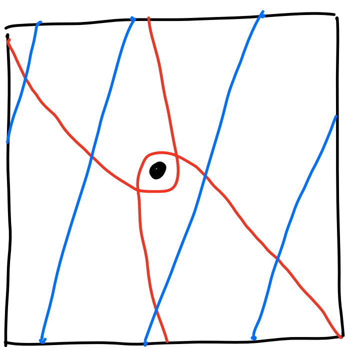


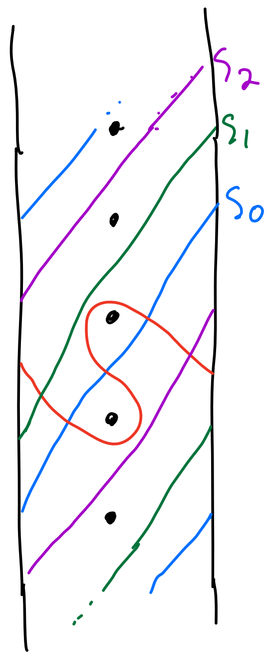
# Surgery formula

**Theorem** (H.): $HF^-(Y^3_{p/q}(K))$ is isomorphic to the Floer homology (in marked torus $T$) of $(\Gamma, b)$ with $l_{p/q}$.

In $\overline{T}$, need $p$ different lifts of $l_{p/q}$,
this gives spin$^c$ decomposition

e.g. $S^3_3(-T_{2,3})$






| $i$ | $\dim \widehat{HF}(S^3_3(-T_{2,3});\, s_i)$ |
|---|---|
| 0 | 3 |
| 1 | 1 |
| 2 | 1 |

$> 1 \Rightarrow$ NLS

## Invariance

Knot Floer data is an invariant of pair $(Y, K)$

Equivalently, an invariant of pair $(M, \mu)$

3 mfld w/ $\partial M = T^2$ choice of meridian

$\boxed{\text{Conj: Actually does not depend on meridian} \therefore \text{invariant of } M}$

$(\Gamma, b)$ viewed as living in $(\partial M, z)$

Need: Understand how to get knot Floer data for $(M, \mu)$ from knot Floer data for $(M, \mu')$

$\to$ Have for $CFK^-$ (Hedden-Levine)

$\to$ Need for $\Psi$

## Pairing Theorem:

$\varphi: \partial M_1 \to \partial M_2$ or. reversing

Consider $\mathbb{Q}HS^3$ $Y = M_1 \cup_\varphi M_2$

- pick slope $\mu_i$ on each $\partial M_i$ identified by gluing
- Consider $(Y_i, K_i)$ corresponding to $(M_i, \mu_i)$
- $\boxed{\underline{\text{Fact:}}\ Y \cong \text{integer surgery on } K_1 \# K_2 \text{ in } Y_1 \# Y_2}$

To compute $HF^-(Y)$ from knot floer data for $K_i$, need:

- surgery formula ✓
- connected sum formula for knot floer data
  - $CFK(K_1 \# K_2) \cong CFK(K_1) \otimes CFK(K_2)$ ✓ (Ozsváth-Szabó)
  - $\Psi_{K_1 \# K_2} \cong \Psi_{K_1} \otimes \Psi_{K_2}$ ✓ (Zemke)

## Pairing Theorem:

$\varphi: \partial M_1 \to \partial M_2$ or. reversing

Consider $\mathbb{Q}HS^3$ $Y = M_1 \cup_\varphi M_2$

$\mu_2 = \varphi(\mu_1)$, $(M_i, \mu_i) \leftrightarrow (Y_i, K_i)$

**Theorem** (H.) If $(\Gamma_i, b_i) \subset (\partial M_i, z_i)^{\simeq T}$ is decorated curve representing knot Floer data for $K_i \subset Y_i$, $HF^-(Y)$ isom. to Floer homology $(\text{in}(\partial M_2, z_2))$ of $\varphi(\Gamma_1, b_1)$ and $(\Gamma_2, b_2)$

$$\boxed{\dim \widehat{HF}(Y) = i(\varphi(\Gamma_1), \Gamma_2)}$$

min. intersection number

( $\text{Spin}^c$ decomposition comes from different lifts to $\overline{T}$ )

# END PART I

Key points:

① For $M^3$ with $\partial M = T^2$ $\exists$ invariant $\Gamma_M$,
a htpy class of immersed multicurve in $(\partial M, z)$
↑ marked point

$$\Gamma_M = \coprod_{s \in Spin^c(M)} \Gamma_{M;s}$$

Fact: For any $s$, $[\Gamma_{M_s}] = [\lambda] \in H_1(\partial M)$

② $\dim \widehat{HF}(M_1 \cup_\varphi M_2) = i(\varphi(\Gamma_{M_1}), \Gamma_{M_2})$

lifts to $\overline{T}$ determine spin$^c$ decomposition

# PART II Toroidal L-spaces

Strategy for L-space conjecture for toroidal mfds:

- understand NLS, CTF, LO on pieces
- understand how these properties behave under gluing

**Def** A graph manifold is a 3-mfd for which each piece in JSJ decomposition is Seifert fibered

understand well

Boyer-Rolfsen-Wiest

Lisca-Stipsicz

Boyer-Gordon-Watson

## Boyer-Clay

- defined $\mathcal{D}_*(M)$ = set of $*$-detected slopes
  $* \in \{LO, CTF, NLS\}$
- $\mathcal{D}_{LO} = \mathcal{D}_{CTF} = \mathcal{D}_{NLS}$ for Seif. fibered

**Conj:** $M_i$ 3-mfd with incompressible $\partial M_i = T^2$

$Y = M_1 \cup_\varphi M_2$ is $*$ $\Longleftrightarrow$ $\varphi_*(\mathcal{D}_*(M_1)) \cap \mathcal{D}_*(M_2) \neq \emptyset$

- showed for $M_i$ graph mfds and $* \in \{LO, CTF\}$
  (so LO $\Leftrightarrow$ CTF for graph manifolds)

$\boxed{\text{Goal:}}$ Show for NLS

Special cases: Splicing knot complements (Hedden-Levine, H.)

enough for $\Longrightarrow$ $\left\{ \begin{array}{l} M_i \text{ Floer simple} \\ M_i \text{ simple loop type} \end{array} \right.$ (Rasmussen-Rasmussen) (H.-Watson)

with BC, shows NLS $\Rightarrow$ LO, CTF for graph manifolds

Ozsvath-Szabo + Kazez-Roberts, Bowden shows CTF $\Rightarrow$ NLS

# Detected Slopes

**Defn 1:** $\alpha \in D_{NLS}(M)$ if $M \cup N_2$ is NLS

$\alpha$ glues to longitude of $N_2$

twisted I-bundle over Klein bottle

**Defn 2:** $D_{NLS}(M) = \overline{\{\alpha \mid M(\alpha) \text{ is NLS}\}}$

**Defn 3:** $D_{NLS}(M) = S(M)$

= set of tangent slopes to $\Gamma_M$

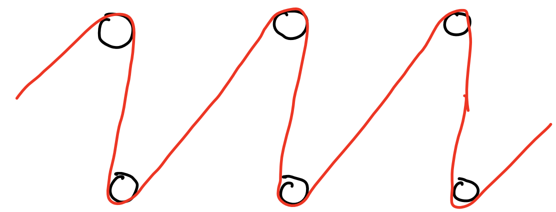
or

all slopes if $\Gamma_M$ has multiple curves in any spin$^c$ str or a non-primitive curve

set of slopes tangent to <u>any</u> representative of $\Gamma_M$

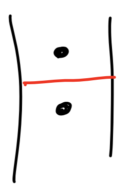
## Pegboard position

we will always "pull curves tight"



think: curve is elastic rope
radius $\epsilon$ peg at marked point

- Both curves pulled tight $\Longrightarrow$ minimal position
- Helps determine $S(M)$:
  each corner determines interval of tangent slopes,
  converges to closed interval as $\epsilon \longrightarrow 0$
- **Lemma:** $M$ is bdy compressible $\Longleftrightarrow$ $\Gamma_M$ has no corners when pulled tight

eg. $M = D^2 \times S^1$, $\Gamma_M =$ 

(pf uses that CFK detects genus / Thurston norm)

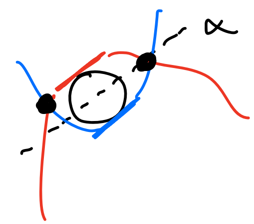
**Theorem:** $M_1 \cup_\varphi M_2$ is NLS $\iff$ $\exists\, \alpha$ with $\alpha \in S(M_1)$, $\varphi(\alpha) \in M_2$

($\partial M_i$ incompressible)

- clear that both sides are true if either $\Gamma_{M_1}$ or $\Gamma_{M_2}$ have non-primitive curve or $>1$ curve for some spin$^c$ str.

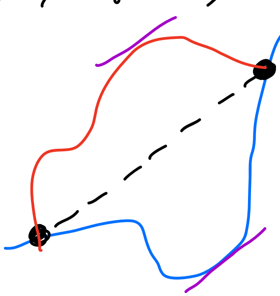
  so may assume one primitive curve per spin$^c$ str.

- "$\Leftarrow$" look at $\alpha$ tangency at peg (must have one on each side of peg)




both intersections on same lift $\Rightarrow$ same spin$^c$ str $\Rightarrow$ NLS

- "$\Rightarrow$" NLS $\Rightarrow$ $\exists$ lifts of curves in $\varphi(\Gamma_{M_1})$ and $\Gamma_2$ that intersect twice




Let $\alpha$ be slope of segment connecting points

By Mean Value Theorem, $\alpha$ is tangent slope to both curves

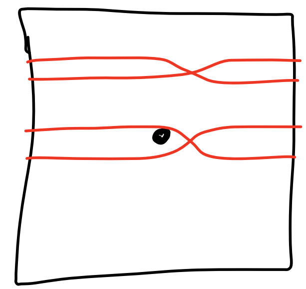
//

Remains to relate different definitions of $D_{NLS}(M)$

Prop: $D_{NLS}(M) = S(M)$

$N_2$ def'n

Key fact: immersed curve for $N_2$ is



$S(N_2) = \{\lambda\}$

rational longitude

pf

Gluing theorem $\Longrightarrow$

$$\underset{\alpha \leftrightarrow \lambda}{M \cup N_2} \text{ is NLS} \iff \alpha \in S(M)$$

$$\Updownarrow$$

$$\alpha \in D_{NLS}(M)$$