Intro. to HF (Heegaard Floer)

Lecture 1: 3-mfld. HF - applications, formal structure, examples

Lecture 2: 3-mfld. HF - sketch of construction

Lecture 3: 3-mfld. HF - sketch of construction

Lecture 4: HF for knots (HFK) - applications, formal structure, examples

Lecture 5: HF for knots (HFK) - sketch of construction, surgery formula

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Applications of 3-mfld. HF:

Q. about 3-mflds.

- **L-space conjecture**

  I.e., HF sees left-orderability, taut foliations (conjecturally)

- **Thurston norm**

  Given a class $x \in H_2(Y; \mathbb{Z})$, what is the minimal genus of a smooth representative of x?

- Surgery questions

  Every 3-mfld. is Dehn surgery on a link L in S^3.

  Given Y, what is the minimal # of components of L which are needed?

Q. about mflds. between 3-mflds. ('relative' 4-mfld theory)

- Definite bounding

  Every (orientable) 3-mfld bounds a (smooth) (orientable) 4-mfld.

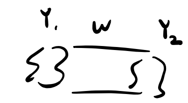
  What kind of 4-mflds. does a given Y bound? — { constraints on the intersection form; positive definite, negative definite, etc.

- Homology cobordism

  Every $\mathbb{Z}HS^3$ bounds a topological $\mathbb{Z}HB^4$ (in fact contractible mfld.)

  Which $\mathbb{Z}HS^3$ bound smooth $\mathbb{Z}HB^4$?

Def. Y_1 and Y_2 ($\mathbb{Z}HS^3$) are <u>homology cobordant</u> if $\exists$ a $\mathbb{Z}H(S^3 \times I)$ cobordism W b/w them.



homologically a cylinder

$$\Theta^3_{\mathbb{Z}} = \{\text{all } \mathbb{Z}HS^3\} / \text{hom. cob.}$$

How big is $\Theta^3_{\mathbb{Z}}$? Does it have torsion?

Formal structure and examples:

<u>$\widehat{HF}(Y)$</u>: <u>a vector space over $\mathbb{F}$</u> — $\mathbb{Z}/2\mathbb{Z}$

Add'l properties:

$$\widehat{HF}(Y) = \bigoplus_{\substack{s \text{ a spin}^c\text{-} \\ \text{structure on } Y}} \widehat{HF}(Y, s)$$

(these are in (non-canonical) bijection w/ $H^2(Y;\mathbb{Z}) = H_1(Y;\mathbb{Z})$)

If s is torsion, then may put an absolute grading (by a coset of $\mathbb{Z}$ in $\mathbb{Q}$) on $\widehat{HF}(Y, s)$

<u>Ex.</u>

$$\widehat{HF}(S^3) = \mathbb{F}_0$$

$$\widehat{HF}(\Sigma(2,3,5)) = \mathbb{F}_2$$

$$\widehat{HF}(\Sigma(2,3,7)) = \mathbb{F}_0^2 \oplus \mathbb{F}_{-1}$$

<u>Ex.</u> $\widehat{HF}(L(p,q), s) = \mathbb{F}$ for each s (p spin^c structures)

<u>Ex.</u> $\widehat{HF}(S^1 \times S^2, s) = \begin{cases} \mathbb{F}_{\frac{1}{2}} \oplus \mathbb{F}_{-\frac{1}{2}} & s = 0 \\ 0 & \text{else} \end{cases}$ ($\mathbb{Z}$ - spin^c structures)

$$\left(\Sigma(p,q,r) = \{(x,y,z) \in \mathbb{C}^3 / x^p + y^q + z^r = 0\} \cap S^5_\varepsilon \right)$$
$$\gcd(p,q,r) = 1$$

<u>Fact</u> If Y is a $\mathbb{Z}HS^3$, have a $\mathbb{Z}$-grading.

<u>Fact</u> If s is torsion, $\dim \widehat{HF}(Y,s) \geq 1$

We will mostly focus on $\mathbb{Q}HS^3$, so each s is torsion.

Def. A $\mathbb{Q}HS^3$ Y is an __L-space__ if $|\widehat{HF}(Y)| = |H_1(Y;\mathbb{Z})|$ (equiv., if $\widehat{HF}$ is minimal). [lens spaces, $\Sigma(2,3,5)$ are examples]

L-space conjecture: can tell a lot about Y by checking whether $\widehat{HF}$ is trivial!

> Aside involving the Thurston norm:
>
> **Def.** Let $S = \cup S_i$ be an embedded surface in Y. Let
>
> $$\chi_-(S) = \sum_i \max\{\underbrace{-\chi(S_i)}_{2g-2}, 0\}$$
>
> For $\phi \in H_2(X;\mathbb{Z})$ $(= H^1(X;\mathbb{Z}))$, let
>
> $$t(\phi) = \min_{[S]=\phi} \{\chi_-(S)\}$$
>
> **Thm.** (OS) $t(\phi) = \min\{\, |\langle c_1(\mathfrak{s}) \cup \phi, [Y]\rangle| \text{ s.t. } \widehat{HF}(Y,\mathfrak{s}) \neq 0 \,\}$

__$HF^-(Y)$__: __a module over $\mathbb{F}[u]$__.

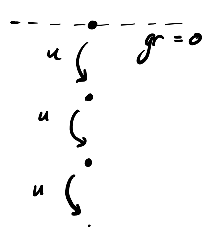
[Also have HF^∞ and HF^+, which we will discuss next lecture. These are determined by HF^-.]

Again,

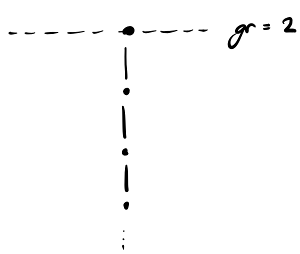
$$HF^-(Y) = \bigoplus_{\mathfrak{s}} HF^-(Y,\mathfrak{s}) \qquad (\deg u = -2)$$

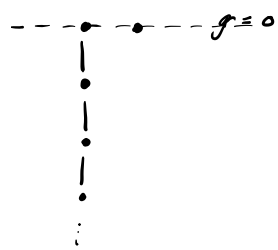
If $\mathfrak{s}$ is torsion, then may put an absolute grading on $HF^-(Y,\mathfrak{s})$

Ex. $HF^-(S^3) = \mathbb{F}[u]_0$ $\qquad HF^-(\Sigma(2,3,5)) = \mathbb{F}[u]_2$ $\qquad HF^-(\Sigma(2,3,7)) = \mathbb{F}[u]_0 \oplus \mathbb{F}_0$



"infinite (downwards) tower"



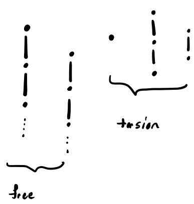


Ex. $HF^-(L(p,q), s)$ = a shifted $\mathbb{F}[U]$ for each s.

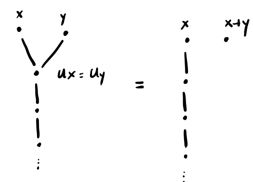
Fact Any $\mathbb{F}[U]$-module is isomorphic to

$$\underbrace{\left(\bigoplus_i \mathbb{F}[U]\right)}_{\text{free part}} \oplus \underbrace{\left(\bigoplus_j \mathbb{F}[U]/U^{n_j}\right)}_{U\text{-torsion part}}$$

(summands might be grading-shifted!)



Ex.



Exercise 1: Prove this. (Think of similar thm. for abelian gps.)

Fact If Y is a $\mathbb{Q}HS^3$, then $HF^-(Y,s)$ has exactly one nontorsion tower for each s.

Def. $HF_{red}(Y,s)$ = torsion part of $HF^-(Y,s)$

Def. A $\mathbb{Q}HS^3$ is an L-space if $HF_{red}(Y,s) = 0$ for each s. (Will see this is equivalent to prev. def.)

Aside involving cobordisms.

Def. For simplicity, let Y be a $\mathbb{Z}HS^3$.

$$d(Y) = \text{grading of top of "the" } U\text{-nontorsion tower in } HF^-(Y)$$

Thm. (OS) If Y bounds a -ve definite mfld. then $d(Y) \geq 0$.

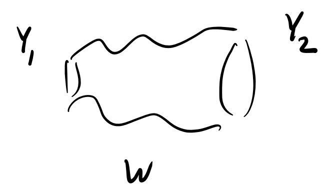
" +ve definite mfld. then $d(Y) \leq 0$.

" $\mathbb{Z}HB^4$, then $d(Y) = 0$.

Thm. (OS) $d(Y_1 \# Y_2) = d(Y_1) + d(Y_2)$

$d(-Y) = -d(Y)$

Cobordism maps



Let W be a cobordism from Y_1 to Y_2.

Each spin^c- structure s on W gives a map: $Y_1 \hookrightarrow W \hookleftarrow Y_2$

$$F_{W,s} : HF^-(Y_1, s|_{Y_1}) \longrightarrow HF^-(Y_2, s|_{Y_2})$$ $H^2(Y_1;\mathbb{Z}) \leftarrow H^2(W;\mathbb{Z}) \rightarrow H^2(Y_2;\mathbb{Z})$

$F_{W,s}$ is $\mathbb{F}[U]$-linear $\left(F_{W,s}(U \cdot x) = U \cdot F_{W,s}(x) \right)$

Fact $F_{W,s}$ sees basic properties of W.

1) $F_{W,s}$ does not preserve grading (in general), but has a homogenous grading shift

$$\Delta_{W,s} = \frac{c_1(s)^2 - 2\tilde{\chi}(W) - 3\sigma(W)}{4}$$

2) If W is negative-definite, then $F_{W,s}$ takes U-nontorsion elements to U-nontorsion elements (in particular, $F_{W,s}$ of a U-nontorsion element is $\neq 0$)

Exercise 2 - Prove that if Y_1 and Y_2 are homology cobordant, then $d(Y_1) = d(Y_2)$.
[Note that if W is a homology cob., then is has only one spin^c- structure and $\Delta_{W,s} = 0$.]