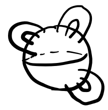
Lecture 2 - Construction of HF (Heegaard splittings and intro.)

Heegaard splittings

Let Y be a 3-mfld. A Heegaard splitting of Y is a splitting

$$Y = H_1 \cup_\partial H_2$$

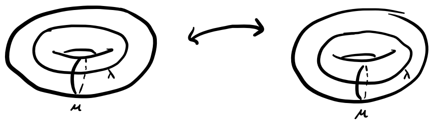
into two handlebodies glued via some bdry. diffeo.



a genus-3 handlebody

Ex. $S^3 = B^3 \cup B^3$

Ex.



$\left.\begin{matrix}\mu \leftrightarrow \mu \\ \lambda \leftrightarrow \lambda\end{matrix}\right\} \Rightarrow S^1 \times S^2$ $\qquad$ $\left.\begin{matrix}\mu \leftrightarrow \lambda \\ \lambda \leftrightarrow \mu\end{matrix}\right\} \Rightarrow S^3$

Exercise 1: Convince yourself of this.

E.g. $S^3 = \partial B^4 = \partial(B^2 \times B^2)$

$= (\partial B^2) \times B^2 \cup B^2 \times (\partial B^2)$

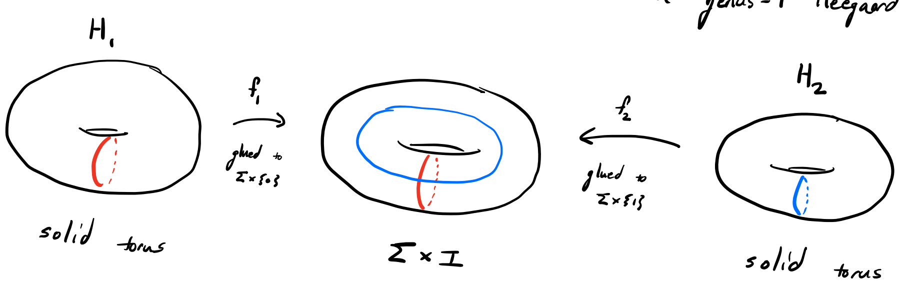
$= S^1 \times B^2 \cup B^2 \times S^1$

Note that most of the subtlety comes from the boundary diffeo.

Fact - every 3-mfld admits a Heegaard splitting

Slightly more abstract way of thinking about Heegaard splittings:

start w/ the "middle surface" Σ. Let's think about a genus-1 Heegaard splitting:



For the construction of Y, it suffices to record the image of the red meridian under f_1 and the blue meridian under f_2. To see this, think of decomposing H_1 into a disk and a ball:

Can think of gluing these two parts separately. Once we glue in the thickened disk, the remaining gluing is "uninteresting".

Exercise 2: Make this precise.

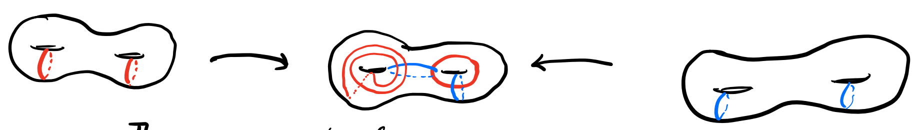
Def. A <u>Heegaard diagram</u> (of genus g) is a closed, oriented surface Σ_g, together with:

1) g simple, closed, disjoint curves
   $\alpha_1, \cdots, \alpha_g$ ("α curves")
   s.t. $\alpha_1, \cdots, \alpha_g$ are lin. ind. in $H_1(\Sigma_g, \mathbb{Z})$

2) "
   $\beta_1, \cdots, \beta_g$ ("β curves")
   "

$(\Sigma_g, \bar{\alpha}, \bar{\beta})$

(note: the α and β curves may intersect each other.

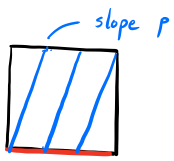


The α_g are thought of as the image of the curves on the left.

Likewise for the β_g.

Exercise 3 - Verify that attaching thickened disks along $\alpha_1 - \alpha_g$ results in boundary S^2 iff) $\alpha_1, \cdots, \alpha_g$ are lin. ind. in $H_1(Y, \mathbb{Z})$.

Exercise 4 - Verify that



Is a Heegaard diagram for $L(p,1)$.

Exercise 5 - Find a Heegaard splitting / diagram for $T^3 = S^1 \times S^1 \times S^1$

Fact - a 3-mfld Y will admit many different Heegaard splittings.

However, any two splittings are related by a sequence of <u>Heegaard moves</u>.

We will discuss this later.

Defining $\widehat{CF}$ — chain complex underlying $\widehat{HF}$

Let Y be a 3-mfld. and $(\Sigma_g, \bar{\alpha}, \bar{\beta})$ be a Heegaard diagram for Y.

Consider

$$\mathrm{Sym}^g(\Sigma_g) = (\Sigma_g \times \cdots \times \Sigma_g) \Big/ \begin{smallmatrix}\text{action of}\\ \text{permutation gp. } S_g\end{smallmatrix} \qquad \left(= \begin{smallmatrix}\text{unordered } g\text{-tuples}\\ \text{of points in } \Sigma_g\end{smallmatrix} \right)$$

This is a $2g$-dim'l manifold. (Locally looks like $\mathrm{Sym}^g(\mathbb{C})$. To see $\mathrm{Sym}^g(\mathbb{C})$ is a mfld, send $\underset{(\alpha_i \in \mathbb{C})}{(\alpha_1, \ldots, \alpha_g)} \longrightarrow \underbrace{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_g)}_{\text{coeff. allow us to view } \in \mathbb{C}^g}$.)

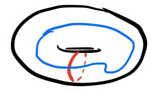
Inside $\mathrm{Sym}^g(\Sigma_g)$, we have the <u>totally real tori</u>

$\mathbb{T}_\alpha = (\alpha_1 \times \cdots \times \alpha_g)/S_g$ (= unordered g-tuples of points in Σ_g w/ each on a distinct α_i)

$\mathbb{T}_\beta = (\beta_1 \times \cdots \times \beta_g)/S_g$ (= " β_i)

These are g-dim'l submanifolds of $\mathrm{Sym}^g(\Sigma_g)$.

<u>Ex.</u> a genus-1 Heegaard diagram is straightforward



$\mathrm{Sym}^g(\Sigma_g) = T^2$

$\mathbb{T}_\alpha = \alpha$

$\mathbb{T}_\beta = \beta$

<u>Ex.</u> a genus-2 Heegaard diagram ?

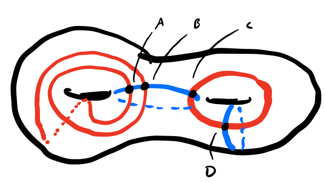
$\mathrm{Sym}^2(\Sigma_2) = T^4 \# \overline{\mathbb{C}P}^2$ — <u>Exercise</u> 8 - Prove this using the Abel-Jacobi map. (Rather difficult.)

$\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are two T^2's inside $\mathrm{Sym}^2(\Sigma_2)$.

<u>Def.</u> The chain complex $\widehat{CF}$ is spanned (over $\mathbb{F}$) by the intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ (under certain genericity conditions).

Explicitly, these are g-tuples $(x_1, \ldots, x_g) \in \mathrm{Sym}^g(\Sigma_g)$ s.t. each x_i is on a distinct α_i and each x_i is on a distinct β_i.

<u>Ex.</u>

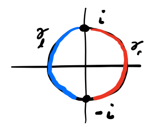


generators :
(A, D) and (B, D)

To define the differential ∂, we must discuss <u>Whitney disks</u>. This will consume the rest of the lecture.

Def. A *Whitney disk* from $\overline{x}$ to $\overline{y}$ $(\overline{x},\overline{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta)$ is a continuous map

$$\phi : \mathbb{D} \longrightarrow \mathrm{Sym}^g(\Sigma_g)$$

$\mathbb{D} =$ 

$\mathbb{C}$

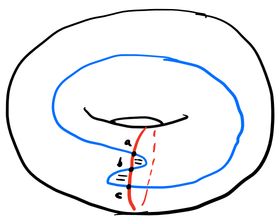
s.t.

1) $\phi(-i) = \overline{x}$ and $\phi(i) = \overline{y}$

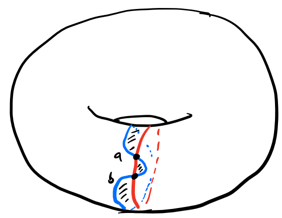
2) $\phi(e_r) \subset \mathbb{T}_\alpha$ and $\phi(e_\ell) \subset \mathbb{T}_\beta$.

Let $\pi(\overline{x},\overline{y})$ be the set of homotopy classes of disks from $\overline{x}$ to $\overline{y}$.

Ex.

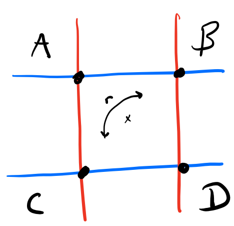


$\pi_2(c,b)$
$\pi_2(a,b)$



$\pi_2(a,b)$

Ex. Say we are in a genus-2 surface

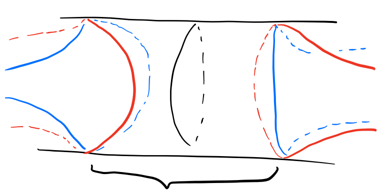


$\overline{x} = (A, D)$ and $\overline{y} = (B, C)$ are generators

there is an arc $\subset \mathbb{T}_\alpha$ from $\overline{x}$ to $\overline{y}$
there is an arc $\subset \mathbb{T}_\beta$ from $\overline{x}$ to $\overline{y}$

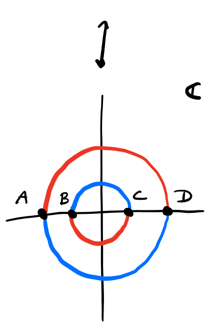
Exercise 6 - show $\{(x, r(x)) \mid x \in \text{the above square}\} \subset \mathrm{Sym}^2(\Sigma_2)$ is a disk from $\overline{x}$ to $\overline{y}$. (Think about branched double cover.)

Ex. Say we are in a genus-2 surface



(A, B)
(A, C)
(D, B)
(D, C)

are generators



Exercise 7 - show $\{(z, \frac{1}{z}) \mid z \in \text{annulus}\} \subset \mathrm{Sym}^2(\Sigma_2)$ is a disk from $\overline{x} = (A,B)$ to $\overline{y} = (C,D)$

Whitney disks and partitioning $\widehat{CF}$:

It turns out that the elements of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are partitioned into $|H_1(Y, \mathbb{Z})|$ - many equivalence classes. s.t. if $\bar{x}$ and $\bar{y}$ are in different equivalence classes, then automatically $\pi_2(\bar{x}, \bar{y}) = \emptyset$ for homological reasons.

What does this mean?

Let $\bar{x}, \bar{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Choose arcs $a \subset \mathbb{T}_\alpha$, $b \subset \mathbb{T}_\beta$

$$\partial a = \bar{y} - \bar{x}$$
$$\partial b = \bar{y} - \bar{x}$$

Then $a-b$ form an element of $H_1(\mathrm{Sym}^g(\Sigma_g))$. If this is $\neq 0$, then clearly there are no disks w/ bdry $a-b$.

But, perhaps we simply chose the wrong a and b. Any other arc $a' \subset \mathbb{T}_\alpha$ from $\bar{x}$ to $\bar{y}$ differs from a by an elt. of $i_*(H_1(\mathbb{T}_\alpha))$. (Similarly for b.)

Thus we should look at the class of $a-b$ in the quotient

$$H_1(\mathrm{Sym}^g(\Sigma_g)) / i_*(H_1(\mathbb{T}_\alpha)) \oplus i_*(H_1(\mathbb{T}_\beta))$$

If this is still $\neq 0$, then clearly $\pi_2(\bar{x}, \bar{y}) = \emptyset$.

Exercise 8 - Show that

$$H_1(\mathrm{Sym}^g(\Sigma_g)) / i_*(H_1(\mathbb{T}_\alpha)) \oplus i_*(H_1(\mathbb{T}_\beta))$$
$$\cong H_1(\Sigma_g) / \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g \rangle$$
$$\cong H_1(Y).$$