Lecture 3 – Defining HF

The situation so far:

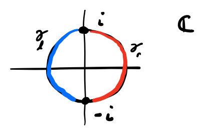
Let Y be a 3-mfld. and $(\Sigma_g, \bar{\alpha}, \bar{\beta})$ be a Heegaard diagram for Y. Consider:

$$\underset{\alpha_1 \times \cdots \times \alpha_g}{\overset{\shortparallel}{\mathbb{T}_\alpha}} \cap \underset{\beta_1 \times \cdots \times \beta_g}{\overset{\shortparallel}{\mathbb{T}_\beta}} \subset \mathrm{Sym}^g(\Sigma_g)$$

These intersection points will be the generators of $\widehat{CF}$.

Def. A <u>Whitney disk</u> from $\bar{x}$ to $\bar{y}$ $(\bar{x}, \bar{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta)$ is a continuous map

$$\phi : \mathbb{D} \longrightarrow \mathrm{Sym}^g(\Sigma_g)$$

$\mathbb{D} =$ 

s.t.

1) $\phi(-i) = \bar{x}$ and $\phi(i) = \bar{y}$
2) $\phi(\gamma_r) \subset \mathbb{T}_\alpha$ and $\phi(\gamma_l) \subset \mathbb{T}_\beta$.

Let $\pi(\bar{x}, \bar{y})$ be the set of homotopy classes of disks from $\bar{x}$ to $\bar{y}$.

To define ∂, we will need two additional pieces of information:

1) We may give $\mathrm{Sym}^g(\Sigma_g)$ a complex structure.
   (Slight lie. Actually, it is an almost-complex structure.)

2) Choose any point $z \in \Sigma_g - (\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g)$
   $V_z = \{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g) \subset \mathrm{Sym}^g(\Sigma_g)$ — note: V_z is disjoint from $\mathbb{T}_\alpha, \mathbb{T}_\beta$

   This is a codimension-2 submfld. of $\mathrm{Sym}^g(\Sigma_g)$. We call z a choice of <u>basepoint</u>.

Def. A <u>holomorphic</u> Whitney disk (from $\bar{x}$ to $\bar{y}$) is a Whitney disk $\phi : \mathbb{D}^2 \to \mathrm{Sym}^g(\Sigma_g)$ which is holomorphic.

Def. For any homotopy class $\Phi \in \pi_2(\bar{x}, \bar{y})$, define

$$n_z(\Phi) = \text{alg. intersection of } \Phi \text{ w/ } V_z.$$

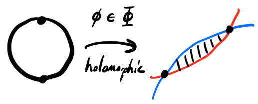
Slogan: "∂ counts holomorphic Whitney disks in $\mathrm{Sym}^g(\Sigma_g)$"

Def. Let $\Phi \in \pi_2(\bar{x}, \bar{y})$. The moduli space of holomorphic rep'n of Φ is

$$\mathcal{M}(\Phi) = \{\phi \in \Phi \text{ s.t. } \phi \text{ is holomorphic}\}$$

Fact Under suitable genericity conditions, this is a (finite-dim'l) manifold.

Note that $\mathcal{M}(\Phi)$ has an $\mathbb{R}$-action.



$\mathbb{R}$ acts by reparametrizing the domain (keeping $\pm i$ fixed)

We can now finally define $\hat{\partial}$ on $\widehat{CF} = \mathrm{span}_{\mathbb{F}} \{\mathbb{T}_\alpha \cap \mathbb{T}_\beta\}$

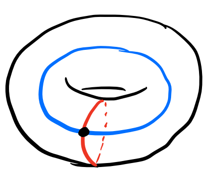
Def.
$$\hat{\partial}\bar{x} = \sum_{\substack{\bar{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ \Phi \in \pi_2(\bar{x},\bar{y}) \text{ s.t.} \\ 1)\ n_z(\Phi) = 0, \text{ and} \\ 2)\ \dim \mathcal{M}(\Phi) = 1}} \left(\# \mathcal{M}(\Phi)/\mathbb{R}\right) \cdot \bar{y}$$
[extend $\mathbb{F}$-linearly]

we will discuss this in a moment.

Thm. (OS)

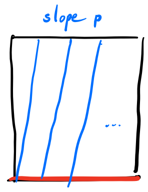
1) $\hat{\partial}$ is a differential on $\widehat{CF}$
2) The chain homotopy type of $(\widehat{CF}, \hat{\partial})$ is independent of all choices (if sufficiently generic), including z.

Ex.



S^3

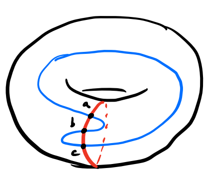
$\widehat{CF} = \mathbb{F}$



$L(p,q)$

$\widehat{CF} = \mathbb{F}^p$

Exercise 1 - Show that in this case, each homotopy class of disk has a unique holomorphic repn. up to $\mathbb{R}$. — True in general for genus-1 heegaard diagram!



S^3

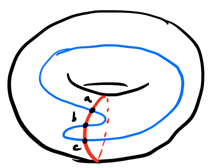
Depending on the placement of z, get:

$\partial a = b$	or	$\partial a = b$	or	$\partial a = 0$
$\partial c = b$		$\partial c = 0$		$\partial c = b$
$\Downarrow$		$\Downarrow$		$\Downarrow$
$\mathbb{F}$		$\mathbb{F}$		$\mathbb{F}$

We now define CF^-.

$CF^- = \operatorname{span}_{\mathbb{F}[u]}\{\mathbb{T}_\alpha \cap \mathbb{T}_\beta\}$ ← for each element in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, introduce an $\mathbb{F}[u]$-tower of generators.

$$\partial^- \bar{x} = \sum_{\substack{\bar{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ \Phi \in \pi_2(\bar{x},\bar{y}) \text{ s.t.} \\ \dim \mathcal{M}(\Phi) = 1}} \#\left(\mathcal{M}(\Phi)/\mathbb{R}\right) \cdot u^{n_z(\Phi)}\, \bar{y} \qquad [\text{extend } \mathbb{F}[u]\text{-linearly}]$$



S^3

Depending on the placement of z, get:

$\partial a = b$, $\partial c = b$ $\Downarrow$ gen. by $a+c$ or $\partial a = ub$, $\partial c = b$ $\Downarrow$ gen. by $a+uc$ or $\partial a = b$, $\partial c = ub$ $\Downarrow$ gen. by $ua+c$

($\mathbb{F}[u]$ in all cases)

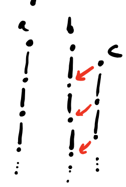
What would be a more interesting complex?

<u>Ex.</u> $CF^- = \operatorname{span}_{\mathbb{F}[u]}\{a, b, c\}$

$\partial^- a = 0$

$\partial^- b = 0$

$\partial^- c = ub$

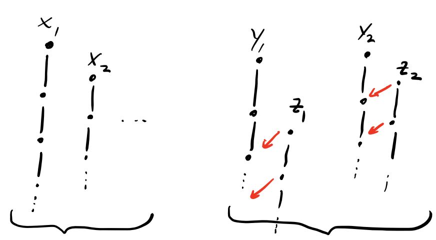


homology is $\Longrightarrow \mathbb{F}[u] \oplus \mathbb{F}$

How to think about CF^- complexes?

In fact, we have a structure theorem for <u>complexes</u> over $\mathbb{F}[u]$.

any complex over $\mathbb{F}[u]$ is homotopy equiv. to



generators that are cycles $\Longrightarrow$ nontorsion towers in homology

"paired" generators $\partial z_i = u^{n_i} y_i$ $\Longrightarrow$ torsion towers in homology

$\widehat{CF}$ may be obtained from CF^- by setting $u = 0$.

Relative gradings and the reason for z. (Sketch)

Given $\overline{x}$ and $\overline{y}$, how would we define the relative grading b/w $\overline{x}$ and $\overline{y}$?

Assume $\overline{x}$ and $\overline{y}$ are in the same spin^c-structure $\Rightarrow$ there is an expected dimension of holomorphic disks from $\overline{x}$ to $\overline{y}$ using the Atiyah-Singer index formula.

Or is there?

Problem: if $g \geq 2$, then in fact there will be many homotopy classes of disk from $\overline{x}$ to $\overline{y}$, and each will have a different expected holomorphic dimension!

Recall $\mathrm{Sym}^2(\Sigma_2) = T^4 \# \overline{\mathbb{CP}}^2$. More generally, can show

$$\pi_2(\mathrm{Sym}^g(\Sigma_g)) = H_2(\mathrm{Sym}^g(\Sigma_g)) = \mathbb{Z}.$$

Exercise 3 Convince yourself that

$$\text{exp. dim.}(\Phi \# S^2) = \text{exp dim.}(\Phi) + 2$$

This seems bad: we can't just define the relative grading by calculating "the" expected dimension! (Although we get a $\mathbb{Z}/2\mathbb{Z}$ grading.)

Key observation: These different moduli spaces will have different n_z-values! (Can show that $V_z = z \times \mathrm{Sym}^{g-1}(\Sigma_g)$ has intersection 1 w/ the generator of $\pi_2(\mathrm{Sym}^g(\Sigma_g))$.)

We use the condition $n_z = 0$ to pick one of these to define the relative grading!