Lecture 4 –

Knot Floer Homology : Applications, formal properties, examples

Q. about knots

1) Distinguishing knots (strictly stronger than $\Delta_K(t)$)

2) Detecting knots — K is uniquely determined by its knot Floer homology in some cases!

[Detects the unknot, trefoil, fig. 8, cinquefoil, 5_2]
$0 \quad 3_1 \quad 4_1 \quad 5_1$

3) Detects Seifert genus

4) Detects fiberedness

Q. about surfaces between knots

$K_1 \sim K_2$ if cobound a smoothly embedded annulus in $S^3 \times I$

1) Sliceness / slice genus / concordance gp.

- How big is $\mathcal{C}$?
- What kind of torsion is in $\mathcal{C}$?
- Is $\mathcal{C}$ generated by various classes of knots?

2) Exotic pairs of (relative) slice surfaces.

- Σ_1, Σ_2 (smooth) slice surfaces for K
- Σ_1 and Σ_2 topologically isotopic but not smoothly isotopic!

$\widehat{HFK}$: a bigraded ($\mathbb{Z} \oplus \mathbb{Z}$) vector space over $\mathbb{F}$

- an Alexander grading
- a Maslov/homological grading

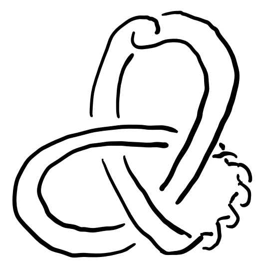
$$\widehat{HFK}(K) = \bigoplus_{\substack{\text{alex.} \\ \text{gradings } s}} \widehat{HFK}(K,s) = \bigoplus_s \left(\bigoplus_i \widehat{HFK}_i(K,s) \right)$$

Ex. $K = U$: $s = 0$: $\mathbb{F}_0$

Ex. $K = 3_1$ (RHT) :
$s = 1$: $\mathbb{F}_2$
$s = 0$: $\mathbb{F}_1$
$s = -1$: $\mathbb{F}_0$

Ex. $K = 4_1$:
$s = 1$: $\mathbb{F}_1$
$s = 0$: $\mathbb{F}_0^3$
$s = -1$: $\mathbb{F}_{-1}$

Ex. $K = Wh_+(T_{2,3})$:
$s = 1$: $\mathbb{F}_{-1}^2 \oplus \mathbb{F}_0^2$
$s = 0$: $\mathbb{F}_{-2}^4 \oplus \mathbb{F}_{-1}^3$
$s = -1$: $\mathbb{F}_{-3}^2 \oplus \mathbb{F}_{-2}^2$



Thm $$\Delta_K(t) = \sum_s \chi\left(\widehat{HFK}(K,s)\right) t^s$$

$$\chi\left(\widehat{HFK}(K,s)\right) = \underbrace{\sum_i (-1)^i \dim \widehat{HFK}_i(K,s)}_{\text{graded Euler characteristic of } \widehat{HFK}(K,s)}$$

Above examples: 1, $t - 1 + t^{-1}$, $-t + 3 - t^{-1}$, 1 !!!

(in fact, any untwisted Wh has $\Delta_K = 1$)

Thm. $\deg(\Delta_K) \leq g_3(K)$

$$\max_s \{ \widehat{HFK}(K,s) \neq 0 \} = g_3(K)$$

Thm. If K is fibered, then Δ_K is monic.

K is fibered iff $\widehat{HFK}(K, g(K)) = \mathbb{F}$

$HFK^-(K)$: a bigraded $\mathbb{F}[U]$-module

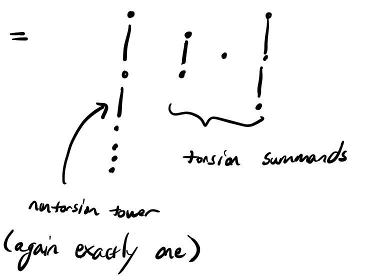
$$HFK^-(K) = \bigoplus_{s \in \mathbb{Z}} HFK^-(K,s)$$

As before, U drops the Maslov grading by 2. Somewhat confusingly, it drops the Alex. grading by 1.
(So the above decomposition isn't really that helpful.)

It's actually more convenient to define new gradings:

$$\begin{cases} gr_u = \text{Maslov grading} \\ gr_v = \text{Maslov grading} - 2 \cdot \text{Alexander grading} \end{cases}$$

Then multiplication by U drops gr_u by 2 and does not change gr_v.

$HFK^- =$ 

[within each tower, have same gr_v]

Ex. $K = U$: $\mathbb{F}[U]_{(0,0)}$

Ex. $K = 3_1$: $\mathbb{F}[U]_{(0,2)} \oplus \mathbb{F}_{(2,0)}$

Ex. $K = Wh_+(3_1)$: $\mathbb{F}[U]_{(0,-2)} \oplus \mathbb{F}^7_{*,*}$

Def. $\tau(K) = -\frac{1}{2} \times$ the gr_v-grading in which $\mathbb{F}[U]$ is located

(always starts at $gr_u = 0$)

Thm. $K_1 \sim K_2 \Rightarrow \tau(K_1) = \tau(K_2)$

Note that $\tau(Wh_+(3_1)) \neq 0$, so $Wh_+(3_1)$ is not smoothly slice!
This is interesting b/c every knot w/ $\Delta_K = 1$ is <u>topologically</u> slice.

Def. $\operatorname{Ord}_u(K) = \min_n \{ U^n \cdot \underbrace{HFK^-_{red}}_{\text{torsion part}} = 0 \}$

Thm. If K is a ribbon knot then any ribbon disk must have at least $\operatorname{Ord}_u(K)$ bands.

Let Σ be a concordance (or even a higher-genus knot cobordism) from K_1 to K_2.
Then we get a cobordism map

$$F_\Sigma : HFK(K_1) \longrightarrow HFK(K_2)$$

<u>Fact</u>: If Σ and Σ' are smoothly isotopic rel boundary, then $F_\Sigma = F_{\Sigma'}$.

This is a potential method for distinguishing surfaces!

- Pick a pair of surfaces Σ, Σ' for K which you know are topologically isotopic.
  [I.e., any pair of slice disks D and D' w/ $\pi_1(B^4 - D) = \pi_1(B^4 - D') = \mathbb{Z}$.]
- Understand enough about F_Σ and $F_{\Sigma'}$ to see that they differ.
