Conj: let M be a closed, orientable, irreducible 3-manifold

$\pi_1(M)$ is left orderable

$\Longleftrightarrow$

M admits a co-orientable taut fol^n

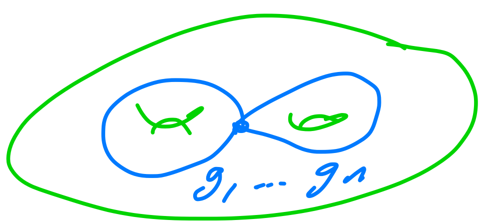
def^n: a foliation is <u>taut</u> if every leaf meets a transverse circle

if a co-orient fol^n $\mathcal{F}$ has a separating leaf then not taut

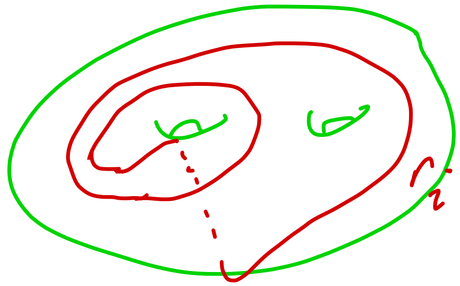
so taut $\Rightarrow$ Reebless

relate question to Heegaard splitting

$\rightsquigarrow$ gives presentation of $\pi_1(M)$



$\pi_1(M) = \langle g_1, \dots, g_n \mid r_1 \dots r_n \rangle$



Th$^{\underline{m}}$ (S. Rasmussen):

Suppose M has Heegaard genus 2 w/ LO $\pi_1(M)$

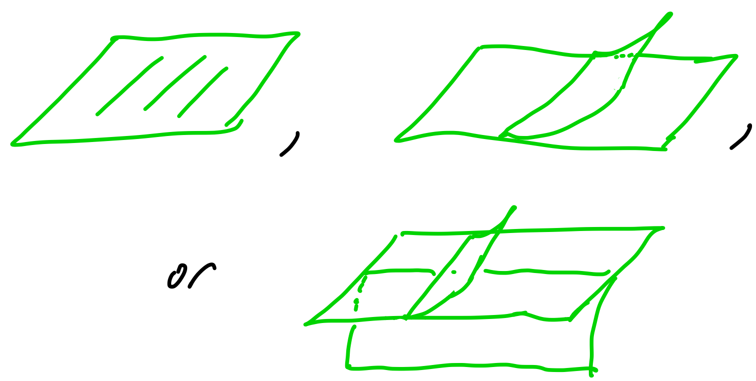
If the group presentation of $\pi_1(M)$ from a genus 2 Heegaard splitting has no subwords that are trivial in $\pi_1(M)$ then M admits a taut fol$^{\underline{n}}$

$\underline{\text{Th}^{\underline{m}}\ (L)}$:

if M has Heegaard genus 2
then left orderable $\Rightarrow$ taut fol$^{\underline{n}}$

tool: branched surface

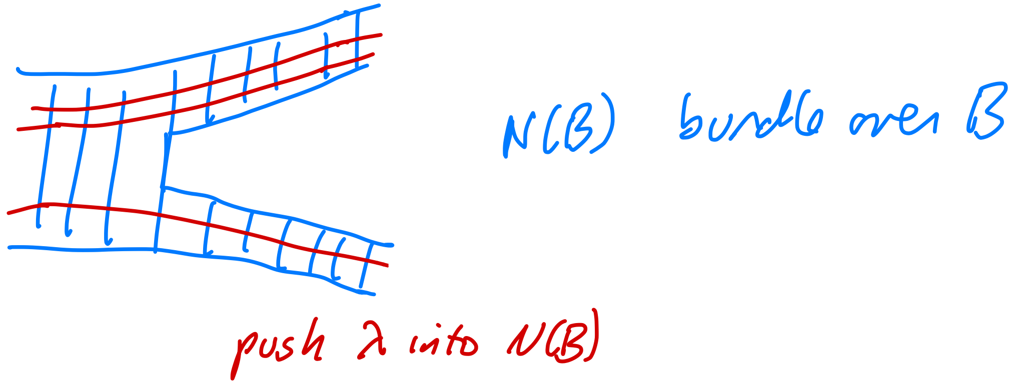
def$^{\underline{n}}$: a branched surface is an object locally modeled on



use these to define <u>laminations</u> in M
(compact foliated submfd)
locally in chart get $\mathbb{R}^2 \times C$

closed set in $\mathbb{R}$

A lamination λ is carried by a branched surface B



if $M \setminus \lambda$ is an I-bundle then λ extend to a foliation

construct a branched surface

genus g Heegaard splitting: $M = H_1 \cup H_2$

S = Heegaard surface

H_1: compressing disks $U_1, U_2, \dots, U_g$

H_2: " " $V_1, V_2, \dots, V_g$

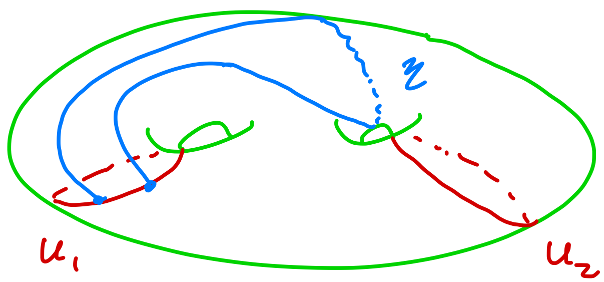
2-complex $\Sigma = (U_1 \cup \dots \cup U_g) \cup S \cup (V_1 \cup \dots \cup V_g)$

let $u_i = \partial U_i$, $v_i = \partial V_i$

assume the Heegaard diagram is "minimal"

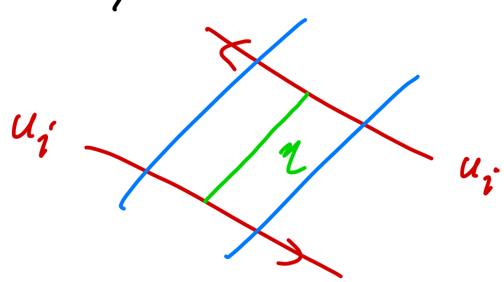
$\nearrow$

<u>no wave</u>



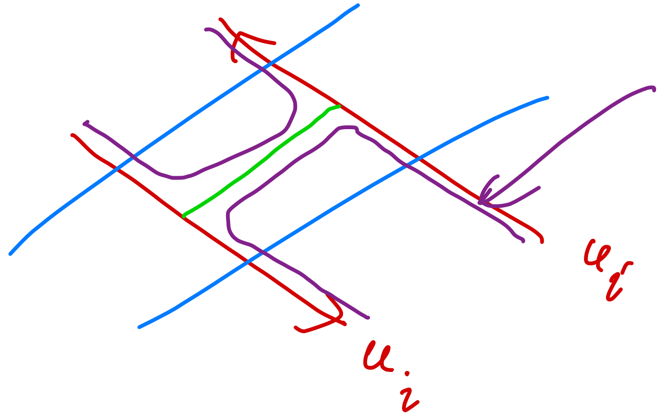
a wave is an arc η that is
nontrivial, and disjoint
from Heegaard Diagram except
$\partial\eta$ on one curve (and on
same side of curve)

example: if you see



get wave η

a <u>wave move</u>

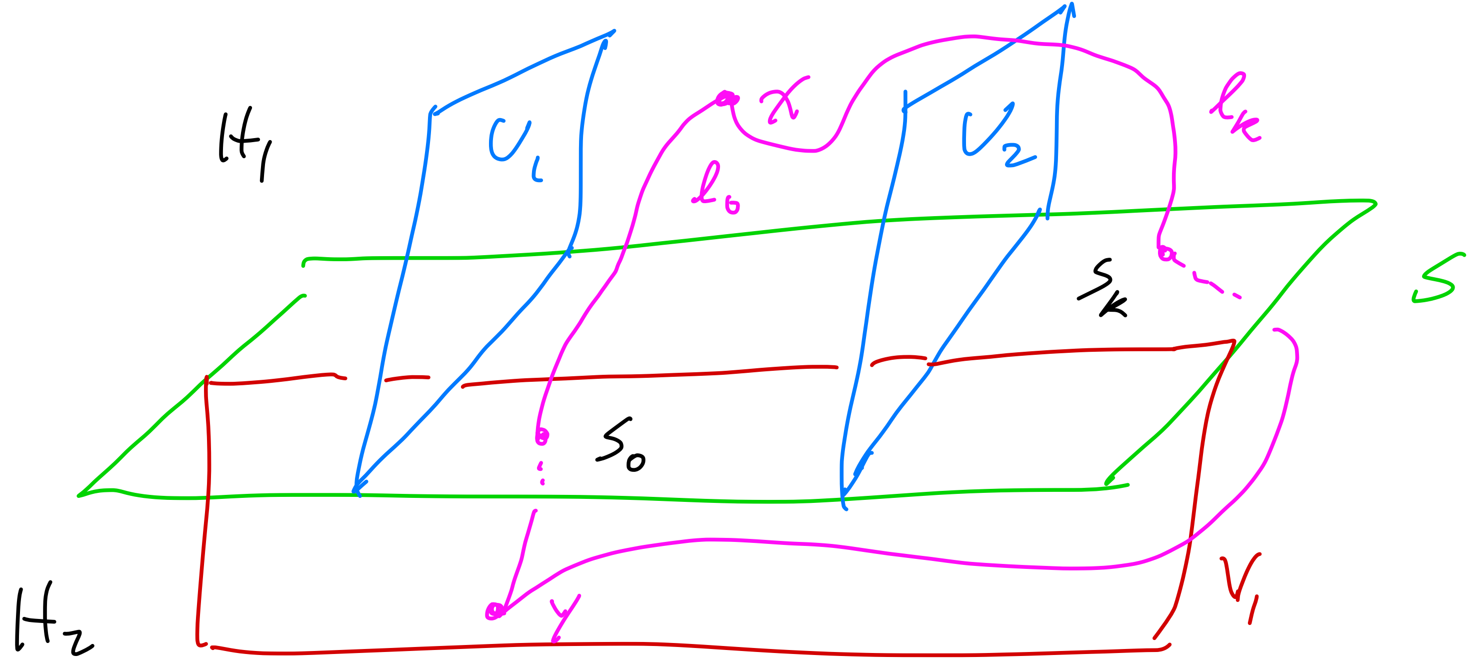


split u_i to 2 curves

replace u_i with u' ← non trivial curve in that not isotopic to another u_j

this simplifies diagram

so assume no wave



pick $x \in H_1 \setminus \bigcup_{i=1}^{g} U_i$

$y \in H_2 \setminus \bigcup_{i=1}^{g} V_i$

$\{u_1, \dots, u_g\}$, $\{v_1, \dots, v_g\}$ cut S into disks

call them $\{S_0, \dots S_m\}$

fix a special disk

let l_i be arcs x to y disjoint from U_i, V_j and $\cap\, S_i$ in one point

get loops $\gamma_i = l_0 * l_i^{-1}$

$$[\gamma_i] \in \pi_1(M, x)$$

clearly $\gamma_0 = 1$

Remark: choice of S_0 does not change the relative order (different choices differ by left mult by fixed elt)

choose S_0 to have minimal order among $\{S_0, \dots S_m\}$

$$\gamma_0 = 1 \qquad \gamma_i \geq 1 \quad \text{for all } i$$

for each U_i get dual loop g_i from x to U_i back to x

each V_i get dual loop $h_i = \gamma_0 * h_i' * \gamma_0^{-1}$

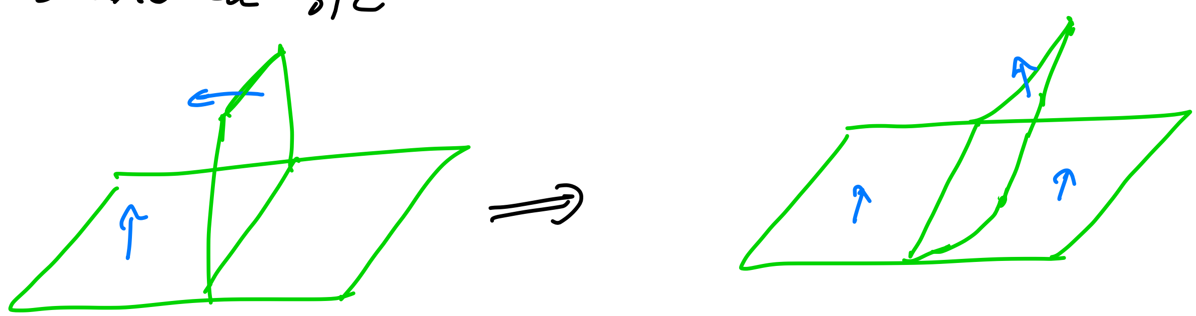
pick an orientation s.t. $g_i, h_i \geq 1$

for disk S_i pick orientation pointing into H_1

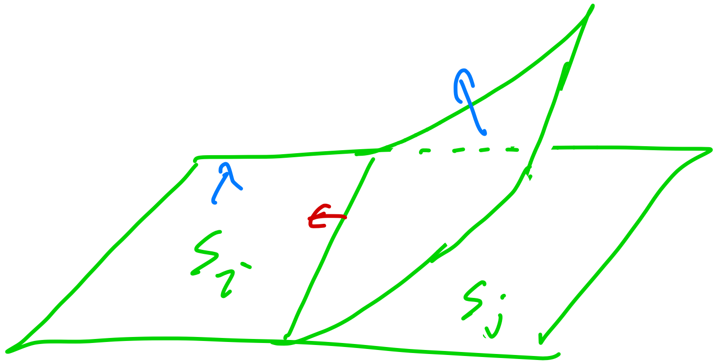
fix orientation on each disk U_i, V_i to agree with or^n on g_i, h_i

now 2-complex can be turned into a branched sfc

eg



take a subset of disks $\{S_0, \ldots, S_k\}$ with associated loops $\gamma_i = 1$



cusp direction

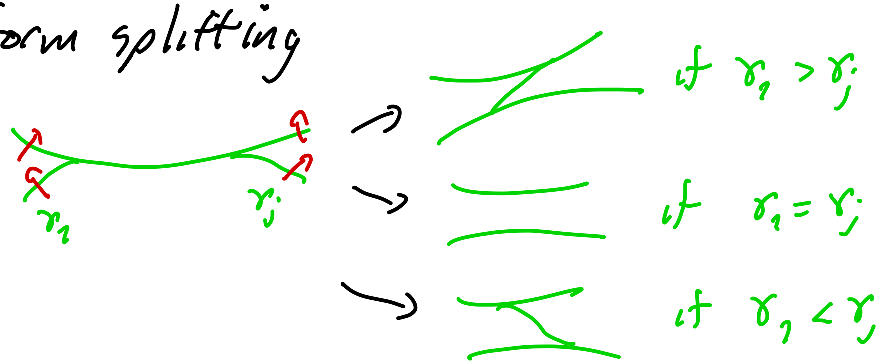
Fact: cusp direction tells us $\gamma_1 > \gamma_j$

so the branch direction at ∂s_j must all point outwards for disks with $\gamma_1 = 1$

get $\hat{B}$ = old branch surface $-$ (interior $(s_0 \cup \dots \cup s_k)$)

Claim: $\hat{B}$ carries a lamination

perform splitting



inverse limit is a lamination

if $g=0$ then lamination extends
to a taut fol$^{\underline{n}}$