ORDERABLE GROUPS AND 3-MANIFOLDS MINICOURSE NOTES

ADAM CLAY

Abstract. This course will begin with an introduction to left-orderable and circularly orderable groups, and will focus on establishing the tools needed to tackle the problem of left-ordering fundamental groups of 3-manifolds–such as the Burns-Hale theorem, actions on $\mathbb{R}$ and S^1, and related Euler class arguments. With these tools in hand, we'll move on to investigating some of the exceptional behaviour of left-orderability with respect to fundamental groups of 3-manifolds. Our chosen examples will draw inspiration from the L-space conjecture, with a focus on manifolds arising from Dehn surgery, and on Seifert fibred spaces. We'll also encounter several open problems along the way, some completely algebraic having only to do with orderable groups, and others that are linked to special cases of the L-space conjecture.

1. Lecture 1: Left-orderings and bi-orderings

Definition 1.1. *A* left-ordering *of a group G is a strict total ordering $<$ of the elements of G such that*

$$g < h \Rightarrow fg < fh$$

for all $f, g, h \in G$. A bi-ordering *of G is a left-ordering that also satisfies*

$$g < h \Rightarrow gf < hf$$

for all $f, g, h \in G$.

A group equipped with a specified left order bi-ordering will be called an *ordered group* and written as a pair $(G, <)$. A group which admits a left-ordering (resp. bi-ordering) will be called a left-orderable group (resp. bi-orderable group). We'll write LO group and BO group for short.

There is an alternative characterisation.

Definition 1.2. *A group G is LO if there exists a subset $P \subset G$ satisfying*

(1) $P \cdot P \subset P$,
(2) $G \setminus \{id\} = P \sqcup P^{-1}$.

A subset P satisfying these two properties is called a positive cone.

There's a correspondence between positive cones and orderings on G via

$$< \ \mapsto \ \{g \in G \mid g > id\}$$

and

$$P \ \mapsto \ g < h \text{ if and only if } g^{-1}h \in P.$$

One can check that this defines a bijection. A group is BO if it admits a $P \subset G$ satisfying (1) and (2) above, and also (3) $gPg^{-1} \subset P$ for all $g \in G$.

Example 1.3. With only the definition in hand, examples are tricky to come by. Obviously $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are all BO groups with equipped with addition. $\square$

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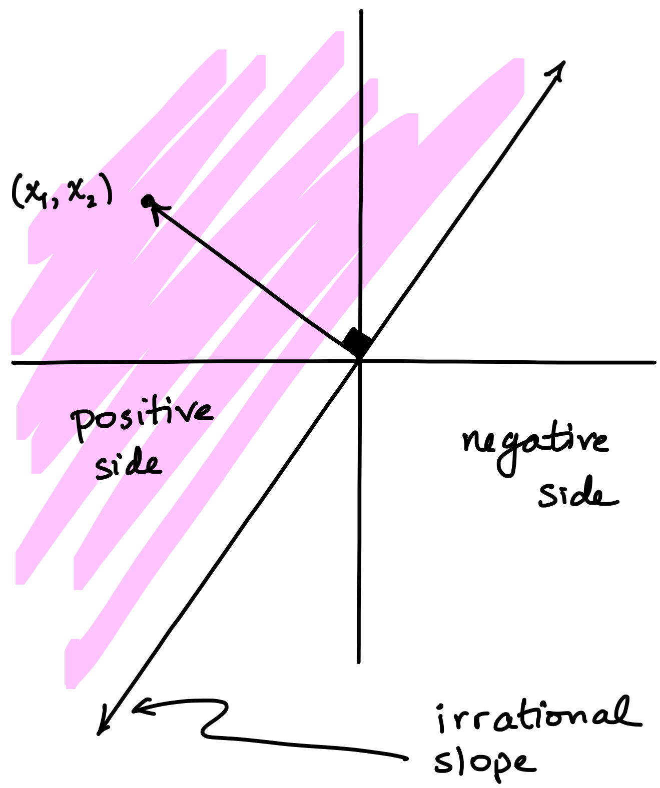


FIGURE 1. Ordering $\mathbb{Z}^2$ using a vector of irrational slope.

Example 1.4. We can order $\mathbb{Z}^2$ by choosing $(x_1, x_2) \in \mathbb{R}^2$ such that x_1/x_2 is irrational, and declaring $(m, n) > (0, 0)$ if and only if $mx_1 + nx_2 > 0$ for all $(m, n) \in \mathbb{Z}^2$. This corresponds to taking a line of irrational slope passing through the origin, and declaring all elements on one side of the line to be the positive cone, see Figure 1. This obviously generalizes to $\mathbb{Z}^n$, though you you have to take care to choose a hyperplane that avoids all the integer lattice points in $\mathbb{R}^n$. $\square$

Getting fancier examples than this without having any sophisticated tools in hand requires a bit of cleverness, so let's see one such example.

Example 1.5. (Due to Magnus, following [15, Chapter 3]) We show in this example that the free group F on countably many generators $\{x_1, x_2, \dots\}$ is bi-orderable, so all of the finitely generated free groups are, too. Set

$$\Lambda = \mathbb{Z}[[X_1, X_2, \dots]],$$

the ring of formal power series in non-commuting variables. Define $\mu : F \to \Lambda$ by

$$\mu(x_i) = 1 + X_i, \text{ and } \mu(x_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \dots.$$

So, for example

$$\mu(x_1 x_2) = (1 + X_1)(1 + X_2) = 1 + X_1 + X_2 + X_1 X_2,$$

or one can check also that, no matter if $p > 0$ or $p < 0$, one always has

$$\mu(x_i^p) = 1 + pX_i + O(2)$$

where $O(2)$ is terms of degree two and higher. Then we observe two lemmas that together complete the proof:

Lemma 1.6. *Let G denote the subgroup of Λ consisting of elements of the form $1 + O(1)$. Then G is bi-orderable.*

Proof. Write the elements of G with lowest degree terms first, and in each degree, order the terms lexicographically (in fact, any fixed ordering in each degree will do). Then, if $U, V \in \Lambda$, declare $U < V$ if the first coefficients where U, V differ satisfy this same inequality. E.g. if

$$U = 1 + X_1 + X_2 + 3X_1^2 + \dots \text{ and } V = 1 + X_1 + X_2 + 5X_1^2 + \dots$$

then $U < V$ because $3 < 5$. From here it is a straightforward check to verify that this works. □

Lemma 1.7. *The homomorphism $\mu : F \to \Lambda$ is injective.*

Proof. One checks that this is true by showing that if $w = x_{i_1}^{n_1} \cdots x_{i_k}^{n_k}$ then the coefficient of the term $X_{i_1}^{n_1} \cdots X_{i_k}^{n_k}$ in the expression for $\mu(w)$ is p, in particular, $\mu(w) \neq 1$. □

We can also create plenty of left-orderable groups using extensions, as this requires little more than the definition.

Proposition 1.8. *Suppose that $P_K \subset K$ and $P_H \subset H$ are positive cones (so that K and H are LO), and that*

$$\{id\} \to K \xrightarrow{i} G \xrightarrow{q} H \to \{id\}$$

is a short exact sequence. Then $P_G = i(P_K) \cup q^{-1}(P_H)$ is a positive cone, in particular, G is LO.

Proof. Check the definition. □

Example 1.9. Torsion-free metabelian groups with torsion-free abelianization are left-orderable. For example, the Heisenberg group over $\mathbb{R}$ is the group of matrices

$$H(F) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \ : \ a, b, c \in \mathbb{R} \right\}$$

is left-orderable for this reason. The group

$$K = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$$

is also left-orderable by the same argument, since there is a short exact sequence

$$\{id\} \to \mathbb{Z} \xrightarrow{i} K \xrightarrow{q} \mathbb{Z} \to \{id\}.$$

Note, however, that the two ends of this short exact sequence are BO groups, while the centre is clearly not BO. □

Proposition 1.10. *If $P_K \subset K$ and $P_H \subset H$ are positive cones of bi-orderings, and*

$$\{id\} \to K \xrightarrow{i} G \xrightarrow{q} H \to \{id\}$$

is a short exact sequence, show that $P_G = i(P_K) \cup q^{-1}(P_H)$ is a positive cone if and only if $gi(P_K)g^{-1} \subset i(P_K)$ for all $g \in G$.

Proof. Check the definition. □

We need additional tools to produce more examples of LO and BO groups, aside from these few. The next theorem characterizes such groups completely. Recall first that a G-action on a set X is *effective* if $g \cdot x = x$ for all $x \in X$ implies $g = id$.

Theorem 1.11. *A group is left-orderable if and only if it admits an effective action by order-preserving bijections on a totally ordered set.*

Proof. If G is LO, first fix a left-ordering $<$ of G and then set $(X,<) = (G,<)$. Then G acts on X by left-multiplication, which is clearly and order-preserving effective action by bijections.

On the other hand, suppose that $(X,<)$ has an effective, order-preserving G-action. Choose a well-order $\prec$ of X (completely unrelated to the ordering $<$ of X!) and for every $g \in G \setminus \{id\}$, set

$$x_g = \min_{\prec}\{x \in X \mid g \cdot x \neq x\}.$$

Note x_g exists because the action is effective. Now, define a positive cone $P \subset G$ by $g \in P$ if and only if $g \cdot x_g > x_g$.

To check this works, it is straightforward to see that $P \sqcup P^{-1} = G \setminus \{id\}$. Then, given $g, h \in P$ suppose that $x_g \prec x_h$, the case of $x_h \prec x_g$ being similar. Observe that $x_{gh} = x_g$, because $h \cdot x_g = x_g$ and so $gh \cdot x_g = g \cdot x_g \neq x_g$; while $g \cdot x = x$ and $h \cdot x = x$ for all $x \prec x_g$. So we compute that

$$gh \cdot x_{gh} = gh \cdot x_g = g \cdot x_g > x_g = x_{gh},$$

so that $gh \in P$. $\square$

Proposition 1.12. *A group G is bi-orderable if and only if it acts effectively by order-preserving bijections on a totally ordered set $(X,<)$, and further*

$$\forall g \in G[(\exists x \text{ s.t. } g \cdot x > x) \Rightarrow (g \cdot x \geq x \text{ for all } x \in X)].$$

When G is a countable group, these results can be greatly improved in a way that connections LO and BO groups to dynamics.

Theorem 1.13. *Suppose that G is countable. Then G is LO if and only if there exists an embedding $G \to \mathrm{Homeo}_+(\mathbb{R})$.*

Proof. The "$\Leftarrow$" direction is already clear from the previous theorem, but "$\Rightarrow$" requires a construction, see e.g. [27].

First, define a *gap* in $(G,<)$ to be a pair of elements (g,h) with $g < h$ such that there is no $f \in G$ with $g < f < h$. Then call an order-preserving embedding $t : (G,<) \to (\mathbb{R},<)$ *tight* if $(a,b) \subset \mathbb{R} \setminus t(G)$ implies that $(a,b) \subset (t(g), t(h))$ for some gap (g,h) in $(G,<)$–i.e., the only gaps in the image of t come from gaps in G.

Tight embeddings exist whenever G is countable, for any ordering $<$ of G. To see this, we enumerate $G = \{g_0 = id, g_1, g_1, \dots\}$ and set $t(id) = 0$. Then if $t(id), t(g_1), \dots, t(g_k)$ are already defined, we set:

$$t(g_{k+1}) = \begin{cases} \max\{t(g_0),\dots,t(g_k)\} + 1 & \text{if } g_{k+1} > \max\{g_0,\dots,g_k\} \\ \min\{t(g_0),\dots,t(g_k)\} - 1 & \text{if } g_{k+1} < \min\{g_0,\dots,g_k\} \\ \dfrac{t(g_j)+t(g_i)}{2} & \begin{array}{l}\text{if } g_j < g_{k+1} < g_i \text{ and} \\ \nexists \ell \in \{0,\dots,k\} \text{ s.t. } g_j < g_\ell < g_i.\end{array} \end{cases}$$

One can verify that this is tight. Then given a tight $t : (G,<) \to (\mathbb{R},<)$ we can build $\rho : G \to \mathrm{Homeo}_+(\mathbb{R})$ via:

(1) If $x \in t(G)$ then $x = t(h)$ for some $h \in G$ and set $\rho(g)(t(h)) = t(gh)$,
(2) if $x \in \overline{t(G)}$, then define $\rho(g)(x)$ so that $\rho(g)$ is continuous on $\overline{t(G)}$, e.g. using sequences,
(3) if $x \in \mathbb{R} \setminus \overline{t(G)}$, then there exists a gap h, k such that $x \in (t(h), t(k))$. Write

$$x = (1-s)t(h) + st(k)$$

for some $s \in (0,1)$ and define

$$\rho(g)(x) = (1-s)t(gh) + st(gk).$$

It is a long check, but this works, and defines a *dynamic realisation* of $(G,<)$. $\square$

One can check that dynamic realisations, as defined in the previous proof, are unique up to conjugation.

Proposition 1.14. *Any two dynamic realisations of $(G,<)$ are conjugate. In other words, given tight embeddings $t,t' : (G,<) \to (\mathbb{R},<)$ and corresponding dynamic realisations $\rho, \rho' : G \to \mathrm{Homeo}_+(\mathbb{R})$, there exists a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that*

$$\rho(g)(x) = h \circ \rho(g) \circ h^{-1}(x)$$

for all $g \in G$ and $x \in \mathbb{R}$.

Remark 1.15. This means that the collection of countable, LO groups is precisely the collection of groups that embed into $\mathrm{Homeo}_+(\mathbb{R})$. Moreover, the study of orderings of such G is equivalent to the study of certain embeddings (up to conjugation) of G into $\mathrm{Homeo}_+(\mathbb{R})$.

Example 1.16. Free groups are left-orderable, because generically, any two homeomorphisms $f, g : \mathbb{R} \to \mathbb{R}$ will generate a free group in $\mathrm{Homeo}_+(\mathbb{R})$.

Example 1.17. The Baumslag-Solitar groups $B(m,n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$ are all left-orderable, and we can exhibit an affine action that demonstrates why in the case where $m = 1$. Define

$$\phi : BS(1,n) \to \mathrm{Homeo}_+(\mathbb{R})$$

by $\phi(a)(x) = nx$ and $\phi(b)(x) = x + 1$ for all $x \in \mathbb{R}$. One can check that this is an embedding, so that $BS(1,n)$ is LO.

In fact, if $n > 0$ then $BS(1,n)$ is BO, but this action does *not* satisfy $\phi(g)(x) > x$ for some $x \in \mathbb{R}$ implies $\phi(g)(x) > x$ for all $x \in \mathbb{R}$, so it's not a dynamic realisation of any bi-ordering. □

These groups are LO, what about non-LO examples? How do they come about?

Example 1.18. Consider the group

$$G = \langle a, b \mid bababa^{-1}b^2a^{-1}, ababab^{-1}a^2b^{-1} \rangle.$$

This group is not LO, but it is torsion-free since it happens to be the fundamental group of a hyperbolic 3-manifold, namely the Weeks manifold. It is torsion-free because the universal cover of W is $\mathbb{H}^3$, which is contractible, so $\pi_1(W)$ has finite cohomological dimension. This implies it cannot have torsion elements.

Rewrite the relations as $b^{-1}ab^{-2}a = (ab)^2 = ba^{-2}ba^{-1}$ and $a^{-1}ba^{-2}b = (ba)^2 = ab^{-2}ab^{-1}$. Now if the group is LO, we can assume $a > 1$. Then we consider cases:

Case 1. $b < id$.

Case 2. $id < a < b$.

Case 3. $id < b < a$. For example in Case 3, $a^{-1}b < 1$. But then the relation $(ba)^2 = a^{-1}ba^{-2}b$ leads to a contradiction, as $(ba)^2$ is positive, whereas $a^{-1}ba^{-2}b = (a^{-1}b)a^{-1}(a^{-1}b)$ must be less than the identity, being a product of three negative elements. □

So far we have given several conditions that a group may satisfy in order to be LO. Next we focus on conditions satisfied by finite subsets of a given group G that can be used to show G is LO.

First, we introduce a property that a semigroup $P \subset G$ can have, which we will call property $(*)$:

For every finite set $\{g_1, \ldots, g_n\} \subset G \setminus \{id\}$, there exist $\epsilon_i = \pm 1$ such that $id \notin sg(P \setminus \{id\}, g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$.

Here, we use $sg(S)$ to denote the subsemigroup of G generated by $S \subset G$.

Theorem 1.19. *([26, Lemma 3.1.1]) Given a semigroup $Q \subset G$, there exists a positive cone $P \subset G$ with $Q \setminus \{id\} \subset P$ if and only if Q satisfies $(*)$.*

Proof. Given $Q \subset G$ with $Q \setminus \{id\}$ contained in some positive cone, it's clear that $(*)$ holds, by choosing ϵ_i so that $g_i^{\epsilon_i} \in P$.

On the other hand, suppose that Q satisfies $(*)$. First observe that if $g \in G \setminus \{id\}$, then one of $sg(Q \setminus \{id\}, g)$ or $sg(Q \setminus \{id\}, g^{-1})$ must satisfy $(*)$. For if not, then there exist $h_1, \ldots, h_n$ and $f_1, \ldots, f_m$ such that

$$id \in sg(Q \setminus \{id\}, g, h_1^{\epsilon_1}, \ldots, h_n^{\epsilon_n})$$

no matter the choice of ϵ_i's, and

$$id \in sg(Q \setminus \{id\}, g^{-1}, f_1^{\nu_1}, \ldots, f_m^{\nu_m})$$

no matter the choice of ν_i's. But then

$$id \in sg(Q \setminus \{id\}, g^{\epsilon}, h_1^{\epsilon_1}, \ldots, h_n^{\epsilon_n}, f_1^{\nu_1}, \ldots, f_m^{\nu_m})$$

no matter the choice of ϵ, ϵ_i's, and ν_i's, contradicting that Q satisfies $(*)$.

So now we set

$$M = \{\text{semigroups } P \subset G \text{ with } Q \subset P \text{ that satisfy } (*)\}.$$

The set M is nonempty, since it contains Q, it is partially ordered by inclusion and one can check that every chain has an upper bound simply by taking unions. So, we can choose $P \in M$ maximal.

Now $(*)$ forces $P \cap P^{-1} \subset \{id\}$, and maximality forces $G \setminus \{id\} \subset P \cup P^{-1}$. So $P \setminus \{id\}$ is the positive cone of a left-ordering of G. $\square$

Corollary 1.20. *A group G is LO if and only if for all $\{g_1, \ldots, g_n\} \subset G \setminus \{id\}$, there exist $\epsilon_i = \pm 1$ such that $id \notin sg(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$.*

Proof. Take $Q = \{id\}$ in the previous theorem. $\square$

Corollary 1.21. *A group G is LO if and only if all of its finitely generated subgroups are LO.*

Corollary 1.22. *All torsion-free abelian groups are BO.*

In fact, it is good enough to consider quotients of all finitely generated subgroups in order to determine whether or not G is LO.

Theorem 1.23 (Burns-Hale, [9])**.** *A group G is LO if and only if for every finitely generated $H \leq G$, there exists a surjective homomorphism $H \to L$ where L is a nontrivial LO group.*

Proof. We apply Corollary 1.20, showing by induction that we can always find the necessary ϵ_i's.

First note that for all $g \in G \setminus \{id\}$, $id \notin sg(g)$ since there exists a surjection $\langle g \rangle \to \mathbb{Z}$.

Now suppose that for all $\{g_1, \ldots, g_n\} \subset G \setminus \{id\}$ with $n \leq k$, there exists $\epsilon_i = \pm 1$ such that $id \notin sg(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$. Consider a collection of elements $\{h_1, \ldots, h_{k+1}\} \subset G \setminus \{id\}$.

Set $H = \langle h_1, \ldots, h_{k+1} \rangle$ and choose $\phi : H \to L$ where L is a nontrivial LO group with positive cone $P_L \subset L$. Assume the h_i's are indexed so that $\phi(h_i) = id$ for $i = 1, \ldots, r$ and $\phi(h_i) \neq id$ for $i = r+1, \ldots, k+1$. Note there is at least one h_i such that $\phi(h_i) \neq id$, since ϕ is a surjection.

Now choose exponents $\epsilon_{r+1}, \ldots, \epsilon_{k+1}$ so that $\phi(h_i) \in P_L$ for $i = r+1, \ldots, k+1$, and by induction, choose $\epsilon_1, \ldots, \epsilon_r$ such that $id \notin sg(h_1^{\epsilon_1}, \ldots, h_r^{\epsilon_r})$.

Given $w \in sg(h_1^{\epsilon_1}, \ldots, h_{k+1}^{\epsilon_{k+1}})$, if w contains any occurences of $h_{r+1}, \ldots, h_{k+1}$ then $\phi(w) \in P_L$ and so $w \neq id$. If w contains no such occurences then $w \in sg(h_1^{\epsilon_1}, \ldots, h_r^{\epsilon_r})$ and $w \neq id$ by induction. Therefore we have found the exponents needed to apply Corollary 1.20. $\square$

We've just spent some time focused solely on LO groups, and it is fair to ask at this point if there are bi-orderability analogs of the ideas above that characterize left-orderability in terms of finite subsets.

Theorem 1.24 (Fuchs [22])**.** *A group G is BO if and only if for all $\{g_1, \ldots, g_n\} \subset G \setminus \{id\}$, there exist $\epsilon_i = \pm 1$ such that $id \notin nsg(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$, where $nsg(S)$ is the normal subsemigroup of G generated by S.*

It's not as straightforward as in the previous case, but this also leads to:

Theorem 1.25. *A group G is BO if and only if every finitely generated subgroup of G is BO.*

This allows us to tidy up a few earlier arguments that depended on the cardinality of certain generating sets:

Corollary 1.26. *All free groups are BO.*

Despite this, there is no BO version of the Burns-Hale theorem.

Example 1.27. The group $K = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$ satisfies: For all finitely generated $H \leq K$, there exists a surjection $H \to \mathbb{Z}$. Despite these maps onto BO quotients, K is clearly not BO. $\square$

With these many techniques, it is perhaps not surprising that LO groups often admit many possible left-orderings. In fact, G only admits finitely many left-orderings when it is a *Tararin group*, meaning G admits a rational series

$$T_0 = \{id\} \triangleleft T_1 \triangleleft \cdots \triangleleft T_{k-1} \triangleleft T_k = G$$

whose quotients T_i/T_{i-1} are rank one abelian, and such that T_i/T_{i-2} is not bi-orderable for any $i = 2, \ldots, k$. For all other groups, there are uncountably many left-orderings.

Bi-orderings behave differently. In this case, there are groups that admit finitely many, countably infinitely many, and uncountably many bi-orderings. However there are no "structure theorems" saying exactly which groups exhibit which kind of behaviour. I.e., the following is open:

Question 1.28. The following questions are open as of the time of writing.

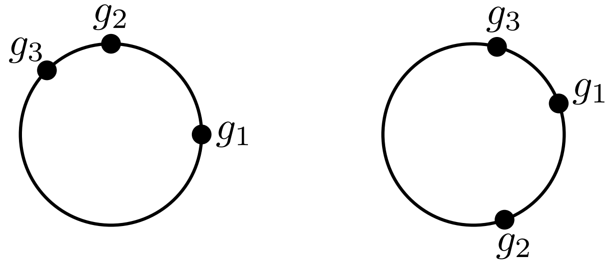
(1) Determine which groups admit finitely many bi-orderings.
(2) Determine which groups admit countably infinitely many bi-orderings.
(3) A nonidentity element $g \in G$ is *generalized torsion* if a product of conjugates of g is equal to the identity. If G is generalized torsion free, must G be LO? (Kourovka Notebook Problem 16.48)

2. Lecture 2: Circular orderings and central extensions

Our goal here will be to introduce circular orderings as an intermediate tool that is to be used when attempting to left-order a group, because this is how they are often used in the study of orderability of 3-manifold groups. First, the following is a way of circular ordering a *set*, which is not yet a group.

Definition 2.1. *A circular ordering c of a set S is a function $c : S^3 \to \{0, \pm 1\}$ such that for all $s_1, s_2, s_3, s_4 \in S$, the following hold:*

(1) *$c^{-1}(0) = \{(s_1, s_2, s_3) \mid s_i = s_j$ for some $i \neq j\}$, and*
(2) *$c(s_2, s_3, s_4) - c(s_1, s_3, s_4) + c(s_1, s_2, s_4) - c(s_1, s_2, s_3) = 0$.*



$c(g_1, g_2, g_3) = 1$ $\qquad$ $c(g_1, g_2, g_3) = -1$

This encodes the notion of triples of points "going in the right direction" around a circle. If we want to (left) circularly order a group, then we do:

Definition 2.2. *A group G is circularly orderable if there is a circular ordering $c : G^3 \to \{0, \pm 1\}$ such that $c(g_1, g_2, g_3) = c(hg_1, hg_2, hg_3)$ for all $h, g_1, g_2, g_3 \in G$.*

You might recognize this as a homogeneous 2-cocycle, which it is, and we will begin to work cohomologically very shortly.

Example 2.3. The groups $\mathbb{Q}/\mathbb{Z}$ and S^1 are circularly orderable in the obvious way. In particular, cyclic groups are all circularly orderable. However $\mathbb{Z}/n\mathbb{Z}$ has $\phi(n)$ circular orderings (just think of the number of embeddings into S^1) whereas $\mathbb{Z}$ has uncountably many circular orderings (again, think of embeddings into S^1).

Example 2.4. Suppose G is LO equipped with an ordering $<$. Define $c_<(g_1, g_2, g_3) = sign(\sigma)$, where $\sigma \in S_3$ is the unique permutation such that $g_{\sigma(1)} < g_{\sigma(2)} < g_{\sigma(3)}$. Check that this works, although schematically it is clear since this is just compactifying the line by adding a point at infinity, to arrive at S^1.

We can also create lexicographic circular orderings, but it doesn't work exactly as one might hope. Here, we need a LO kernel and CO quotient, instead of CO quotient and kernel.

Proposition 2.5. *Suppose that K is a left-orderable group and H is a circularly-orderable group. If there is a short exact sequence*

$$1 \to K \to G \to H \to 1$$

then G is circularly orderable.

Proof. Let c be a circular ordering on H and $<$ a left order on K. Construct a circular ordering c on G as follows (this is the construction from [10]): Let $g_1, g_2, g_3 \in G$ be distinct elements.

(1) If $\phi(g_1), \phi(g_2), \phi(g_3)$ are distinct, $c(g_1, g_2, g_3) = c(\phi(g_1), \phi(g_2), \phi(g_3))$.
(2) If $\phi(g_1) = \phi(g_2)$ and $\phi(g_1) \neq \phi(g_3)$, then $g_2^{-1}g_1 \in K$. If $g_2^{-1}g_1 < id$, set $c(g_1, g_2, g_3) = 1$. If $g_1^{-1}g_2 < id$, set $c(g_1, g_2, g_3) = -1$.
(3) If $\phi(g_1) = \phi(g_2) = \phi(g_3)$, $g_3^{-1}g_1, g_3^{-1}g_2 \in K$. Let $a_1 = g_3^{-1}g_1, a_2 = g_3^{-1}g_2, a_3 = id$. There is a unique permutation $\sigma \in S_3$ such that $a_{\sigma(1)} < a_{\sigma(2)} < a_{\sigma(3)}$. Set $c(g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}) = 1$.

□

There are also plenty of non-circularly orderable groups, but finding them is a bit tricky. I will defer examples of such until later, though I will point out that there are some easy finite examples already available if you are willing to do the work: A finite group is circularly orderable if and only if it is cyclic (this can be proved directly from the definition, the trick is to show that set $G \setminus \{id\}$ becomes totally ordered and that the minimal element in this total ordering generates G). From here, we can also provide a significant source of examples that mirrors what we did in the case of left-orderings.

Theorem 2.6. *A countable group G is circularly orderable if and only if G embeds in* $\mathrm{Homeo}_+(S^1)$.

Proof. Starting with G and building an embedding into $\mathrm{Homeo}_+(S^1)$ can be done similarly to the case of $\mathbb{R}$, or by constructing an embedding of a certain left-orderable lift into $\mathrm{Homeo}_+(\mathbb{R})$, which we will see shortly, and then passing to S^1 by quotienting.

On the other hand, we can show that $\mathrm{Homeo}_+(S^1)$ is circularly orderable as follows. Choose $p \in S^1$ and consider the "short exact sequence"

$$1 \to Stab(p) \to \mathrm{Homeo}_+(S^1) \to \mathrm{Homeo}_+(S^1)/Stab(p) \to 1,$$

where we can identify the left cosets of the stabilizer with the orbit of p. Therefore the left-cosets are circularly orderable in a way that is preserved by left multiplication on the set of cosets, and

$Stab(p)$ is left-orderable since it acts in an order-preserving manner on $S^1 \setminus \{p\} \cong \mathbb{R}$. Modifying the previous proposition to accommodate the fact that the stabilizer is not necessarily normal will give the result. $\square$

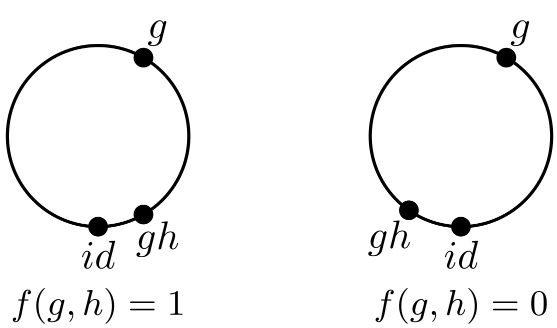
Corollary 2.7. *The group* $\mathrm{PSL}(2, \mathbb{R})$ *is circularly orderable.*

Proof. The action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$ naturally gives an orientation-preserving action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{R}P^1$, thought of as lines through the origin, which is the circle. Or another way: $\mathrm{PSL}(2, \mathbb{R})$ is the group of orientation-preserving isometries of $\mathbb{H}^2$, thought of as unit disk (the Poincaré disk model). This action extends to the boundary of the disk, which is S^1. $\square$

At this point it actually serves our purpose to change to an alternative definition of circular ordering.

Definition 2.8. *A circular ordering on a group G is a function $f : G^2 \to \mathbb{Z}$ satisfying:*

(1) $f(id, g) = f(g, id) = 0$ *for all* $g \in G$,
(2) $f(h, k) - f(gh, k) + f(g, hk) - f(g, h) = 0$ *for all* $g, h, k \in G$,
(3) $f(g, h) \in \{0, 1\}$ *for all* $g, h \in G$,
(4) $f(g, g^{-1}) = 1$ *for all* $g \in G \setminus \{id\}$.



We can abbreviate this by saying that f is an inhomogenous 2-cocycle that additionally satisfies $f(g, h) \in \{0, 1\}$ for all $g, h \in G$ and $f(g, g^{-1}) = 1$ for all $g \in G \setminus \{id\}$, since the first two properties exactly capture what it means to be an inhomogeneous 2-cocycle. Then there's a bijection between circular orderings in terms of c's and circular orderings in terms of f's (e.g. see [13]). Given c, define f via:

$$f^{(c)}(g, h) = \begin{cases} 0 & \text{if } g = id \text{ or } h = id, \\ 1 & \text{if } gh = id \text{ and } g \neq id, \\ \frac{1}{2}(1 - c(id, g, gh)) & \text{otherwise,} \end{cases}$$

and conversely, given f, define c via:

$$c^{(f)}(g_1, g_2, g_3) = \begin{cases} 0 & \text{if } g_i = g_j \text{ for some } i \neq j, \\ 1 - 2f(g_1^{-1}g_2, g_2^{-1}g_3) & \text{otherwise.} \end{cases}$$

This determines a bijection between the set of circular orderings in terms of c's on a group G, and the set of circular orderings written as f's. You're probably wondering why this change matters, it's because we want to investigate circular orderings as cocycles in group cohomology, which in our situation will be easier to do using inhomogeneous cocycles.

To put this all in context, I will need to introduce two objects: The second cohomology group $H^2(G; \mathbb{Z})$ of a group G, and the set of equivalence classes of central extensions of G by $\mathbb{Z}$, denoted $\mathcal{E}(G; \mathbb{Z})$ (for more details and context, see [8, Chapter 4]).

We set:

$$Z^2(G;\mathbb{Z}) = \{f : G^2 \to \mathbb{Z} \mid f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2), \text{ and } f(id, g) = f(g, id) = 0\}$$

and

$$B^2(G;\mathbb{Z}) = \{f : G^2 \to \mathbb{Z} \mid \exists h : G \to \mathbb{Z} \text{ with } f(g_1, g_2) = h(g_1) + h(g_2) - h(g_1g_2) \text{ and } h(id) = 0\},$$

and then set

$$H^2(G;\mathbb{Z}) = Z^2(G;\mathbb{Z})/B^2(G;\mathbb{Z}).$$

This is the second cohomology group of G with integer coefficients (and trivial action on $\mathbb{Z}$). This obviously fits into the context a complete collection of cohomology groups $H^n(G;\mathbb{Z})$ defined from collections of inhomogeneous cocycles, but we need only H^2.

Remark 2.9. Group cohomology can also be defined by using $K(\pi, 1)$'s (Eilenberg-Mac Lane spaces), that is, topological spaces with prescribed fundamental group π and trivial higher homotopy groups. Then the cohomology of the group G is defined to be the singular cohomology of the space $K(G, 1)$.

Next, suppose that we have central extensions

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_1} H_1 \xrightarrow{q_1} G \to 1$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_2} H_2 \xrightarrow{q_2} G \to 1.$$

We say that these extensions are equivalent if there exists a homomorphism $\phi : H_1 \to H_2$ such that $\phi \circ i_1 = i_2$ and $q_2 \circ \phi = q_1$, and these conditions force ϕ to be an isomorphism. The set of equivalence classes of central extensions is $\mathcal{E}(G;\mathbb{Z})$.

There's a bijection $H^2(G;\mathbb{Z}) \iff \mathcal{E}(G;\mathbb{Z})$, sending $id \in H^2(G;\mathbb{Z})$ to the trivial extension $G\times\mathbb{Z}$, and it works like this:

Starting with a cocycle: Given a inhomogeneous cocycle $f : G^2 \to \mathbb{Z}$, define a central extension $\widetilde{G}_f$ by taking the set $G\times\mathbb{Z}$ and equipping it with the multiplication $(g, a)(h, b) = (gh, a+b+f(g, h))$. Then there's a central copy of $\mathbb{Z}$ generated by $(id, 1)$ and an obvious quotient onto the original group G, making $\widetilde{G}_f$ the required central extension.

Starting with a central extension:. Suppose we have

$$1 \longrightarrow \mathbb{Z} \xrightarrow{i} H \xrightarrow{q} G \to 1,$$

and choose a function $s : G \to H$ with $s(id) = id$ and $q\circ s(g) = g$ for all $g \in G$. I.e. s is a normalized section. Define a function $f : G^2 \to \mathbb{Z}$ by capturing the defect of s from being a homomorphism, i.e. $f(g, h)$ is the element in $i(\mathbb{Z})$ given by $s(g)s(h)s(gh)^{-1}$. We need to check this is a cocycle, but it will work out.

Fair question: If this classical machinery is applied to circular orderings, which are themselves cocycles (i.e. elements of $Z^2(G;\mathbb{Z})$), then what is special about the corresponding central extension?

Proposition 2.10. *([30]) If G is a group and $f : G^2 \to \{0, 1\}$ is a circular ordering, then $\widetilde{G}_f$ is a left-ordered group.*

Proof. We need only define a positive cone. Recalling that the elements of $\widetilde{G}_f$ are pairs $(g, a) \in G \times \mathbb{Z}$, the positive cone is:

$$P = \{(g, a) \mid n \geq 0\} \setminus \{(id, 0)\}.$$

To check that $P \cdot P \subset P$ is easy, just apply the multiplication $(g, a)(h, b) = (gh, a + b + f(g, h))$ and observe that $a, b \geq 0$ implies $a + b + f(g, h) \geq 0$ since f takes values in $\{0, 1\}$.

To see that $P \sqcup P^{-1} = \widetilde{G}_f \setminus \{(id, 0)\}$, we observe that for all $g \in G \setminus \{id\}$, we have$(g, a)^{-1} = (g^{-1}, -a-1)$, since

$$(g, a)(g^{-1}, -a-1) = (id, a - a - 1 + f(g, g^{-1})) = (id, 0);$$

and for $(0, a)$ the inverse is $(0, -a)$. In particular $(g, 0)^{-1} = (g^{-1}, -1)$ when $g \neq id$, which is usually the case that causes some concern for most people's intuition. In any event, we conclude $P \cap P^{-1} = \emptyset$, with $P \cup P^{-1} = \widetilde{G}_f \setminus \{(id, 0)\}$ being obvious. □

We've now reached our main goal:

Corollary 2.11. *Suppose that G is a group with circular ordering $f : G^2 \to \{0, 1\}$ such that $[f] = id \in H^2(G; \mathbb{Z})$. Then G is left-orderable.*

Proof. Since $[f] = id \in H^2(G; \mathbb{Z})$ and our correspondence between $H^2(G; \mathbb{Z})$ and $\mathcal{E}(G; \mathbb{Z})$ maps the identity to the equivalence class of the trivial extension, we know that $\widetilde{G}_f$ is isomorphic to the group $G \times \mathbb{Z}$. In particular, since $\widetilde{G}_f$ is left-orderable, so is $G \times \mathbb{Z}$, so G is left-orderable. □

Remark 2.12. . We can be a little more careful and improve these observations:

(1) We can actually do better than asserting that there's an abstract isomorphism of $\widetilde{G}_f$ with $G \times \mathbb{Z}$. Since $[f] = id$, there's a map $\eta : G \to \mathbb{Z}$ satisfying $\eta(id) = 0$ and $f(g, h) = \eta(g) + \eta(h) - \eta(gh)$ for all $g, h \in G$. Then $\psi : \widetilde{G}_f \to G \times \mathbb{Z}$ given by $\psi(g, a) = (g, a + \eta(g))$ is an explicit isomorphism.
(2) While this tells us that G is left-orderable, it's not telling us that f itself is a left-ordering. What I mean here is: Recall that given a LO $<$ of G, we set $c_<(g_1, g_2, g_3) = sign(\sigma)$, where $\sigma \in S_3$ is the unique permutation such that $g_{\sigma(1)} < g_{\sigma(2)} < g_{\sigma(3)}$. We could equally rewrite this as $f_<$ for a inhomogeneous cocycle f, and call such an f a "secret left-ordering" because, well, secretly it's just a left-ordering. This theorem does *not* imply that f is a secret left-ordering.

To detect secret left-orderings, we need something more delicate. We can define

$$Z_b^2(G; \mathbb{Z}) = \{f \in Z^2(G; \mathbb{Z}) | f \text{ is a bounded function}\},$$

and

$$B_b^2(G; \mathbb{Z}) = \{f \in B^2(G; \mathbb{Z}) | f \text{ is a bounded function}\},$$

and then set

$$H^2(G; \mathbb{Z}) = Z_b^2(G; \mathbb{Z}) / B_b^2(G; \mathbb{Z}).$$

This defines the second *bounded* cohomology group. The next proposition and its proof come from [2], though the result first appeared in a dynamical language in [23].

Proposition 2.13. *A circular ordering $f : G^2 \to \{0, 1\}$ is a secret left-ordering if and only if $[f] = id \in H_b^2(G; \mathbb{Z})$.*

Proof. Suppose f is a secret left ordering. Let P be the positive cone of the secret left ordering $<$ corresponding to f. Define a bounded function $d : G \to \mathbb{Z}$ by

$$d(g) = \begin{cases} 0 & \text{if } g \in P \cup \{id\} \\ 1 & \text{if } g \in P^{-1}. \end{cases}$$

Then for all $g, h \in G$, $ff(g, h) = d(g) - d(gh) + d(h)$. Therefore $[f] = id \in H_b^2(G; \mathbb{Z})$.

Conversely, suppose that for all $g, h \in G$, $ff(g, h) = d(g) - d(gh) + d(h)$ for some bounded function $d : G \to \mathbb{Z}$, and recall $f(g, h) \in \{0, 1\}$. We first show $d(g) \in \{0, 1\}$ for all $g \in G$.

Suppose $d(g) < 0$ for some $g \in G$. Then $d(g) + d(g^{k-1}) - d(g^k) = f(g, g^k) \geq 0$ and via induction on k we may conclude that $d(g^k) \leq -k$ for all $k \in \mathbb{N}$, contradicting the boundedness of d. Similarly, if $d(g) \geq 2$, induction on k with the fact that $d(g) + d(g^{k-1}) - d(g^k) \leq 1$ allows us to conclude $d(g^k) \geq 1 + k$ for all $k \in \mathbb{N}$. Therefore $d(g) \in \{0, 1\}$ for all $g \in G$.

Now define $P \subset G$ by $g \in P \cup \{id\}$ if and only if $d(g) = 0$. We check that P is a positive cone. Given $g, h \in P$, note $d(g) + d(h) - d(gh) \in \{0, 1\}$. Therefore we must have $d(gh) = 0$ so $P \cdot P \subset P$. Let $g \in G \setminus \{id\}$. If $d(g) = 0$, then $d(g) + d(g^{-1}) = f_c(g, g^{-1}) = 1$. Therefore $d(g^{-1}) = 1$. Similarly if $d(g) = 1$, $d(g^{-1}) = 0$. Therefore $G = P \sqcup \{id\} \sqcup P^{-1}$ and P is a positive cone on G.

Let $<$ be the left ordering on G with positive cone P. It suffices to show $g_1 < g_2 < g_3$ implies $c^f(g_1, g_2, g_3) = 1$. If $g_1 < g_2 < g_3$, then $d(g_1^{-1}g_2) = d(g_1^{-1}g_3) = d(g_2^{-1}g_3) = 0$. We can recover c^f from f via the prescription

$$c^f(g_1, g_2, g_3) = \begin{cases} 0 & \text{if } g_i = g_j \text{ for some } i \neq j \\ 1 - 2f(g_1^{-1}g_2, g_2^{-1}g_3) & \text{otherwise.} \end{cases}$$

Since g_1, g_2, and g_3 are distinct we have

$$c^f(g_1, g_2, g_3) = 1 - 2f(g_1^{-1}g_2, g_2^{-1}g_3) = 1 - 2(d(g_1^{-1}g_2) + d(g_2^{-1}g_3) - d(g_1^{-1}g_3)) = 1$$

completing the proof. $\square$

With all this, we're able to arrive at a testable criterion for circular orderability of a group.

Theorem 2.14. *([1, Theorem 2.6]) Suppose that f is a circular ordering of G such that $[f] \in H^2(G; \mathbb{Z})$ has order k. Then G contains a left-orderable normal subgroup H such that $G/H \cong \mathbb{Z}/k\mathbb{Z}$.*

Proof. Consider the cocycle $kf : G^2 \to \{0, k\}$ defined by taking k times the inhomogeneous cocycle f. Since we assume $[f]$ has order k, we know that $[kf] = id \in H^2(G; \mathbb{Z})$, and thus there exists a map $\eta : G \to \mathbb{Z}$ satisfying $\eta(id) = 0$ and $kf(g, h) = \eta(g) - \eta(gh) + \eta(h)$ for all $g, h \in G$. Therefore if $q_k : \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ is the quotient then $q_k \circ \eta : G \to \mathbb{Z}/k\mathbb{Z}$ is a homomorphism, since the defect of η from being a homomorphism is $kf(g, h)$, an element divisible by k. One can check that this map is surjective since (if not) this would force $[f]$ to be of finite order less than k.

Now we determine the kernel. Note that there's an injective map $\phi : \widetilde{G}_f \to \widetilde{G}_{kf}$ given by $\phi(g, a) = (g, ka)$, and since $[kf] = id$, there's an isomorphism $\psi : \widetilde{G}_{kf} \to G \times \mathbb{Z}$ given by $\psi(g, a) = (g, a + \eta(g))$.

On the other hand, the kernel of $q_k \circ \eta$ is precisely the subgroup

$$H = \{g \in G \mid \eta(g) \text{ is divisible by } k\ \}$$

of G, which is obviously isomorphic to the subgroup

$$H' = \{(g, 0) \mid \exists a \in \mathbb{Z} \text{ such that } ka + \eta(g) = 0\} \leq G \times \mathbb{Z}.$$

But this last subgroup is inside the subgroup $\psi \circ \phi(\widetilde{G}_f)$, which is a left-orderable subgroup since $\psi \circ \phi$ is injective and $\widetilde{G}_f$ is LO. So H' and thus H is LO. $\square$

This finally gives a way to construct interesting non-circularly orderable groups, which we illustrate by example.

Example 2.15. Consider the group

$$G = \langle a, b \mid bababa^{-1}b^2a^{-1}, ababab^{-1}a^2b^{-1} \rangle.$$

from the first lecture, the fundamental group of the Weeks manifold W, which is the hyperbolic 3-manifold of smallest volume. It satisfies $H^2(G; \mathbb{Z}) \cong H^2(W; \mathbb{Z}) \cong H_1(W; \mathbb{Z}) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$, where the first isomorphism is because W is a $K(\pi, 1)$ and the second isomorphism is Poincaré duality. In particular, every nontrivial element of $H^2(G; \mathbb{Z})$ is of order five. Supposing there is a circular ordering f of G, we consider two possibilities:

(1) $[f] = id \in H^2(W; \mathbb{Z})$. This would mean that G is left-orderable, which it is not (see first lecture).

(2) $[f] \neq id \in H^2(W;\mathbb{Z})$ and has order 5. Then G must have an index 5 normal LO subgroup. This is a computational problem that we can attack by computer, and Calegari-Dunfield did just that in [11]. They showed all index 5 normal subgroups are not LO, so G is not CO.

Remark 2.16. Why pass to f instead of using c?

The reasons are twofold. First, describing central extensions is easier using inhomogeneous cocycles. Also surprisingly, $[c]$ and $[f]$ are different elements in H^2 if you apply the standard "change of coordinates" to pass from homogeneous to inhomogeneous cocycles. We find $[c] = 2[f_c]$, so which one is "correct"?

There is another natural way to construct a left-orderable central extension of a circularly ordered group G, if the group is countable. You can first use the circular ordering c to build an embedding $\rho : G \to \mathrm{Homeo}_+(S^1)$, then take the preimage in $\widetilde{\mathrm{Homeo}}_+(S^1)$ to arrive a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ (hence a LO group) that is a $\mathbb{Z}$-central extension of G. If we look at the element of $H^2(G;\mathbb{Z})$ corresponding to this central extension, it's not $[c]$, it's $[f_c]$!

Lecture 3: Ordering 3-manifold groups

All our 3-manifolds will be compact and connected unless otherwise specified, and they'll always be orientable/oriented to keep things simple. Some theorems below have non-orientable counterparts. We initiate the study by reducing the problem to studying *prime* manifolds. Recall that a n-manifold M other than S^m is called *prime* if, whenever $M \cong M_1 \# M_2$ then one of M_1 or M_2 is homeomorphic to S^n. Here, $\#$ is the connect sum of M_1 and M_2. This is defined by choosing n-balls $B_i \subset int(M_i)$ for $i = 1, 2$ with sufficiently nice boundary (i.e. the boundary sphere has a product neighbourhood) and a homeomorphism $h : \partial B_1 \to \partial B_2$ and then gluing $M = M_1 \setminus (int(B_1)) \cup_h M_2 \setminus (int(B_2))$.

Theorem 2.17. *Suppose that M is a compact, connected, oriented 3-manifold other than S^3. Then there exist prime manifolds $M_1, \dots, M_n$ that are unique up to permutation of their indices and orientation-preserving homeomorphism such that*

$$M = M_1 \# \dots \# M_n.$$

We call the decomposition $M_1 \# \dots \# M_n$ the prime decomposition. By the Seifert-Van Kampen theorem, for a manifold M as in the previous theorem,

$$\pi_1(M) \cong \pi_1(M_1) * \dots * \pi_1(M_n).$$

So we need:

Proposition 2.18. *The free product $G * H$ is left-orderable if and only if G and H are both left-orderable. Moreover, if G admits a left-ordering $<_G$ and H admits a left-ordering $<_H$, then $G * H$ admits a left-ordering $<$ whose restriction to G is $<_G$, and whose restriction to H is $<_H$.*

Proof. We present two proofs.

Proof 1: Note there's a map $G * H \to G \times H$ that's induced by the maps $g \mapsto (g, id)$ and $h \mapsto (id, h)$. So there's a short exact sequence

$$\{id\} \to K \to G * H \xrightarrow{q} G \times H \to \{id\},$$

and we can analyze the kernel of this map as follows. Recall that every non-identity element of $G * H$ can be uniquely written as a product

$$w = a_1 a_2 a_3 \dots a_n$$

where the a_i's are alternately from $G \setminus \{id\}$ and from $H \setminus \{id\}$, we call $\ell(g) = n$ the length of g and set $\ell(id) = 0$. We first note that K is generated by the set

$$S = \{[g, h] = ghg^{-1}h^{-1} \mid g \in G \setminus \{id\} \text{ and } h \in H \setminus \{id\}\}.$$

To see this, we induct on the length of $w \in K$, first noting that if $\ell(gw) = 0$ then w is trivially in $\langle S \rangle$. Now suppose that for $\ell(w) < n$ if $w \in K$ then $w \in \langle S \rangle$, and consider $w \in K$ with $\ell(w) = n$. First we can check that if $w \in K$ then $\ell(w)$ cannot be less than four, and then write

$$w = a_1 a_2 a_3 \dots a_n = (a_1 a_2 a_1^{-1} a_2^{-1})(a_2 a_1 a_3 a_4 \dots a_n) = [a_1, a_2] w'$$

where the a_i's are alternately from $G \setminus \{id\}$ and from $H \setminus \{id\}$. Note that $\ell(w') < n$ since a_1 and a_3 are adjacent and come from the same factor, and so the induction assumption applies, landing w in $\langle S \rangle$.

Next, we can in fact check that S is a free basis, to do this we'll show that no reduced word in S represents id. Write $x_{g,h}$ in place of $[g, h]$ and suppose

$$w = x_{g_1,h_1}^{\epsilon_1} x_{g_2,h_2}^{\epsilon_2} \dots x_{g_n,h_n}^{\epsilon_n}$$

where $g_i \in G \setminus \{id\}$ and $h_i \in H \setminus \{id\}$, $\epsilon_i = \pm 1$ and you never have $(g_i, h_i) = (g_{i+1}, h_{i+1})$ and $\epsilon_i = -\epsilon_{i+1}$ for some $i \in \{1, \dots, n-1\}$. I.e, it's a reduced word in S.

We can prove that W can be written uniquely as an alternating product $a_1 a_2 a_3 \dots a_m$ where the a_i's are alternately from $G \setminus \{id\}$ and from $H \setminus \{id\}$, and either $a_{m-1} a_m = g_n^{-1} h_n^{-1}$ if $\epsilon_n = 1$ or $a_{m-1} a_m = h_n g_n$ if $\epsilon_n = -1$. We prove this by inducting on the length of w in the generators $x_{g,h}$, the case of $n = 1$ being obvious.

Considering only the case $\epsilon_n = 1$ we first apply the induction assumption to $x_{g_1,h_1}^{\epsilon_1} x_{g_2,h_2}^{\epsilon_2} \dots x_{g_{n-1},h_{n-1}}^{\epsilon_{n-1}}$ to write w as either

$$w = a_1 a_2 \dots g_{n-1}^{-1} h_{n-1}^{-1} x_{g_n,h_n}^{\epsilon_n} = a_1 a_2 \dots g_{n-1}^{-1} h_{n-1}^{-1} g_n h_n g_n^{-1} h_n^{-1}$$

or

$$w = a_1 a_2 \dots h_{n-1} g_{n-1} g_n h_n g_n^{-1} h_n^{-1}.$$

Note that in the first case, we're done, and in the second case, $h_{n-1} g_{n-1} g_n h_n \neq id$ by our assumption that w is a reduced word in the $x_{g,h}$. So we're done in this case, too. The case of $\epsilon_n = -1$ is similar.

Now the result follows from

$$\{id\} \to K \to G * H \xrightarrow{q} G \times H \to \{id\},$$

as we can lexicographically order $G \times H$ using given orderings $<_G$ and $<_H$. Then K is free, so it's LO (in fact BO), and so we can lexicographically order $G * H$. Moreover, the ordering on $G * H$ extends $<_G$ and $<_H$. This concludes the first proof.

Proof 2: This is due to Dicks and Sunic [18]. Define a function $\tau : G * H \to \mathbb{Z}$ as follows. First, fix positive cones $P_G \subset G$ and $P_H \subset H$, and given an nonidentity element $w \in G * H \setminus \{id\}$ write it uniquely as

$$w = a_1 a_2 a_3 \dots a_n$$

where the a_i's are alternately from $G \setminus \{id\}$ and from $H \setminus \{id\}$. Define $\eta(w)$ to be 0 if $\ell(w) = 1$, and otherwise:

$$\eta(w) = \begin{cases} 0 & \text{if } a_1, a_n \in G \text{ or } a_1, a_n \in H, \\ 1 & \text{if } a_1 \in G \text{ and } a_n \in H \\ -1 & \text{if } a_1 \in H \text{ and } a_n \in G \end{cases}$$

Recall that $w = a_1 a_2 a_3 \dots a_n$ and define

$$\tau(w) = |\{i \mid a_i \in P_G \cup P_H\}| - |\{i \mid a_i \in P_G^{-1} \cup P_H^{-1}\}| + \eta(w).$$

One can check that $\tau(w)$ is always odd, so that no $w \in G * H \setminus \{id\}$ satisfies $\tau(w) = 0$. Set

$$P = \{w \in G * H \setminus \{id\} \mid \tau(w) > 0\}.$$

Checking this is a positive cone is a quick case argument, and it obviously extends both P_H and P_G since $\tau(g) = +1$, $\tau(h) = +1$ whenever $g \in P_G$, $h \in P_H$. $\square$

Remark 2.19. Both of the constructions in the previous proof can be generalized to the case of free products with arbitrarily many factors.

Therefore

Theorem 2.20. *Suppose that M is a compact, connected, oriented 3-manifold other than S^3 and that*

$$M = M_1 \# \ldots \# M_n$$

is its prime decomposition. Then $\pi_1(M)$ is LO if and only if $\pi_1(M_i)$ is LO for all $i = 1, \ldots, n$.

So we need only investigate prime 3-manifolds. A 3-manifold M is irreducible if every embedded 2-sphere in M bounds an embedded 3-ball in M. This is closely related to primeness, as the only prime, reducible orientable manifold is $S^1 \times S^2$. (To show irreducible implies prime is just by definition, for the other direction, consider separating and non-separating spheres. If non-separating then you get $(S^1 \times S^2) \setminus int(B^3)$ in M, so you can write it as $(S^1 \times S^2)\#M_0$ for some M_0. But primeness says $M_0 \cong S^3$.)

So this means that

Theorem 2.21. *With M as above, $\pi_1(M)$ is LO if and only if the irreducible factors in its prime decomposition have LO fundamental group.*

So in fact we need only consider irreducible 3-manifolds in our investigation of left-orderability of $\pi_1(M)$. To go further, we need two theorems:

Theorem 2.22. [24, Theorem 3.15] *Suppose that M is orientable and irreducible, and $\widetilde{M} \to M$ is a covering space. Then $\widetilde{M}$ is irreducible.*

Also the famous "Compact Core Theorem" which allows us to cut down an arbitrary 3-manifold to a smaller, compact submanifold without losing any of the fundamental group.

Theorem 2.23 (Scott, [29])**.** *Suppose that M is a 3-manifold with finitely generated fundamental group. Then there is a compact submanifold $N \subset int(M)$ such that inclusion $i : N \to M$ induces an isomorphism $i_* : \pi_1(N) \to \pi_1(M)$.*

These are the main ingredients we need for the following theorem. We follow the presentation from [15].

Theorem 2.24. [7] *Suppose that M is compact, connected, orientable and irreducible, not S^3. Then $\pi_1(M)$ is left-orderable if and only if there exists a surjective homomorphism from $\pi_1(M)$ onto a nontrivial left-orderable group L.*

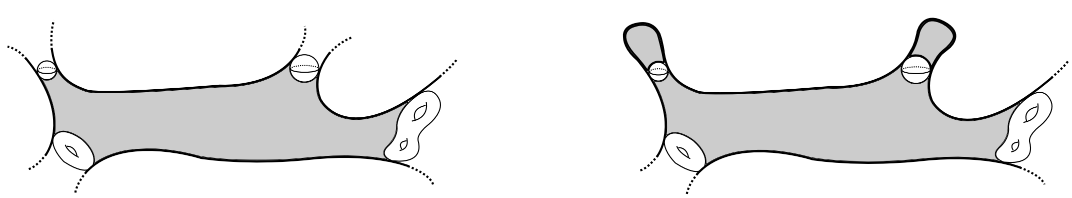
Proof. If $\pi_1(M)$ is LO then just use the identity map. This gives the easy direction.

The difficult direction is to begin with a map $\phi : \pi_1(M) \to L$ and show that $\pi_1(M)$ is LO. We aim to apply the Burns-Hale theorem: If we can show that every finitely generated subgroup of $\pi_1(M)$ has a nontrivial left-orderable quotient, then we're done.

So let $H \subset \pi_1(M)$ be a finitely generated subgroup. There are two cases.

Case 1. The subgroup H has finite index. Then the image of H under the map ϕ is nontrivial and left-orderable, so this case is done.

Case 2. The subgroup H has infinite index. Then there's a covering space $p : \widetilde{M} \to M$ with $p_*(\pi_1(\widetilde{M})) = H$, which is necessarily noncompact but whose fundamental group is finitely generated. By Scott, there's a compact core N for $\widetilde{M}$, and so N must have nonempty boundary. If the boundary of N contains any 2-spheres, we can cap them off with copies of B^3 without changing the fundamental group.



To see this, suppose $S \subset \partial N$ is a 2-sphere, then by irreducibility of $\widetilde{M}$ there's a 3-ball bounded by S in $\widetilde{M}$. But as S separates $\widetilde{M}$, either $N \subset B$ or $B \cap N = S$. The former is impossible since $\pi_1(N)$ is nontrivial, and the latter means we can take $N' = N \cup B^3$ as a new core for $\widetilde{M}$, eliminating S from the boundary.

But now a standard Euler characteristic argument shows that $H_1(N;\mathbb{Z})$ is infinite, since N has nonempty boundary with no 2-sphere components. Here's now. Recall that the Euler characteristic of a closed 3-manifold is always 0, and we create the double $2N$ of N where we glue N to a copy of itself by the identity and observe

$$0 = \chi(2N) = 2\chi(N) - \chi(\partial N).$$

But since $\chi(\partial N) \leq 0$ from our assumptions, then $2\chi(N) = \chi(\partial N) \leq 0$. But $\chi(N)$ is the alternating sum

$$1 - dim(H_1(N;\mathbb{Q})) + dim(H_2(N;\mathbb{Q})) - 0,$$

and if this is going to be ≤ 0 we'll need $dim(H_1(N;\mathbb{Q})) \geq 1$.

So there's a map $\pi_1(\widetilde{M}) \cong \pi_1(N) \to H_1(N) \to \mathbb{Z}$. This finishes the proof. □

Corollary 2.25. *If M is as above and $H_1(M;\mathbb{Z})$ is infinite, then $\pi_1(M)$ is left-orderable.*

Proof. There's the Hurewicz homomorphism $\pi_1(M) \to H_1(M;\mathbb{Z})$, and since $H_1(M;\mathbb{Z})$ is a f.g. infinite abelian group there's a further homomorphism $H_1(M;\mathbb{Z}) \to \mathbb{Z}$. Now apply the previous theorem. □

Remark 2.26. In fact, when $H_1(M)$ is infinite the proof of the previous theorem tells us something more. In this case, every finitely generated subgroup of $\pi_1(M)$ admits a surjection onto $\mathbb{Z}$, not just onto a nontrivial LO group.

This means that with M above having infinite first homology, $\pi_1(M)$ is *locally indicable.* In terms of orderability, this means that $\pi_1(M)$ admits a special kind of left-ordering called a *Conradian* left-ordering. Conradian left-orderings of a group G are left-orderings where $g, h > id$ implies $g^{-1}hg^2 > id$ for all $g, h \in G$. While *a priori* this seems like a bizarre artificial condition, it turns out to be extremely useful and natural if one does a "deep dive" into orderability. This is the class of orders that sits naturally between left-orderings and bi-orderings, as in:

$$\{ \text{ bi-orderings of } G \} \subset \{ \text{ Conradian orderings of } G \} \subset \{ \text{ left-orderings of } G\}.$$

Corollary 2.27. *Suppose that M is as above, then every nontrivial infinite-index subgroup of $\pi_1(M)$ is LO.*

Proof. We just repeat the proof of Case 2 in the proof of the previous theorem, noting that we didn't need ϕ for that part of the theorem. □

We can also come up with a circular orderability version of this theorem. Recall that a *cyclic cover* is by definition a regular covering space whose deck transformation group is cyclic.

Theorem 2.28. *Suppose that M is as above (compact, connected, orientable and irreducible) and has infinite fundamental group. Then $\pi_1(M)$ is circularly orderable if and only if there exists a surjective homomorphism from $\pi_1(M)$ onto an infinite circularly orderable group.*

Proof. If $\pi_1(M)$ is CO, then just use the identity map. This is the easy direction.

Supposing there exists such a surjection onto an infinite circularly orderable group C, we get a corresponding short exact sequence

$$1 \to K \to \pi_1(M) \to C \to 1$$

where K is left-orderable by the previous corollary. It follows that $\pi_1(M)$ is lexicographically circularly orderable. □

We need the next result to fully investigate circular orderability.

Theorem 2.29. *An irreducible 3-manifold with infinite fundamental group is a $K(\pi, 1)$, i.e. it has trivial higher homotopy groups. In particular, this means that we can calculate the group cohomology $H^2(\pi_1(M); \mathbb{Z})$ from the cohomology of the manifold M, i.e. $H^2(\pi_1(M); \mathbb{Z}) \cong H^2(M; \mathbb{Z})$.*

Proof. First we observe that $\widetilde{M}$ has $\pi_2(\widetilde{M}) \cong \pi_2(M) = 0$ since it is irreducible. This follows from the sphere theorem (which says that if π_2 is nontrivial, then there's a nontrivial element represented by an embedded sphere–but by irreducibility, all such elements are trivial). Next $H_i(\widetilde{M}; \mathbb{Z}) = 0$ for $i \geq 3$ since $\widetilde{M}$ is noncompact and of dimension 3. But now the Hurewicz theorem says that $\pi_i(\widetilde{M}) = 0$ for $i \geq 3$, and so $\pi_i(M) = 0$ for $i \geq 3$ as well. □

We can therefore compute the cohomology of $\pi_1(M)$ from the cohomology of M, when it's irreducible. Now we are ready to show:

Theorem 2.30 ([1])**.** *Suppose that M is a compact, connected, orientable, irreducible 3-manifold with infinite fundamental group. Then $\pi_1(M)$ is circularly orderable if and only if M admits a finite cyclic cover with left-orderable fundamental group.*

Proof. If $H_1(M; \mathbb{Z})$ is infinite, then $\pi_1(M)$ is LO, hence circularly orderable, and so there is nothing to prove in this case (take the trivial cover).

On the other hand, suppose $H_1(M; \mathbb{Z})$ is finite and $\pi_1(M)$ is circularly orderable with circular ordering f. Then $H_1(M; \mathbb{Z}) \cong H^2(M; \mathbb{Z})$ by Poincaré duality and $H^2(M; \mathbb{Z}) \cong H^2(\pi_1(M); \mathbb{Z})$ since M is irreducible. Thus $[f]$ has finite order, say it has order k. Then by Lecture 2 there is a normal, left-orderable subgroup H such that $\pi_1(M)/H$ is cyclic.

On the other hand, if M admits such a cover then we can constuct a lexicographic circular ordering of $\pi_1(M)$ in the obvious way. □

Therefore we have reduced the problem of determining when $\pi_1(M)$ is LO, or circularly orderable, to the case of compact, connected, orientable, irreducible 3-manifolds with finite first homology. This is exactly where we arrive at the L-space conjecture, which says:

Conjecture 2.31. *(The L-space conjecture,* [6, Conjecture 1], [25, Conjecture 5]*) Suppose that M is a compact, connected, orientable irreducible 3-manifold with finite first homology, other than S^3. Then the following are equivalent:*

(1) *M is not a Heegaard Floer homology L-space;*
(2) *M admits a co-orientable taut foliation;*
(3) *$\pi_1(M)$ is left-orderable.*

There's also a CO version of this conjecture, which comes from putting together what we saw so far.

Conjecture 2.32. *(The L-space conjecture, circular orderability version,* [1]*). Suppose that M is a compact, connected, orientable irreducible 3-manifold with finite first homology that is not a lens space. Then the following are equivalent:*

(1) *There exists a finite cyclic cover of M that is not an L-space.*
(2) *There exists a finite cyclic cover of M that supports a co-orientable taut foliation.*
(3) *The fundamental group of M is circularly orderable.*

Let's investigate how to deal with left-orderability of a few of these kinds of manifolds, coming from easy constructions. The first is Seifert fibred manifolds. We recall one characterization of these manifolds, rather than their usual definition:

Theorem 2.33. *(Epstein,* [19]*). If M is a compact 3-manifold with a foliation whose leaves are circles, then M is a Seifert fibre space.*

Here's how to construct some such manifolds, following Hatcher, which we use to understand the fundamental group. We fix:

(1) Σ a compact, connected surface with m boundary components.
(2) disks $D_1, \ldots, D_n \subset \mathrm{int}(\Sigma)$, set $\Sigma' = \Sigma \setminus (\mathrm{int}(D_1) \cup \ldots \cup \mathrm{int}(D_n))$,
(3) $M' = \Sigma' \times S^1$.
(4) For each boundary torus of M', $H_1(T_i; \mathbb{Z}) = H_1(S^1 \times \partial D_i; \mathbb{Z})$ has basis
$$\{[\{1\} \times \partial D_i], [S^1 \times \{pt\}]\} = \{[h_i^*], [h]\}.$$

Choose n reduced fractions $\frac{\beta_i}{\alpha_i} \subset \mathbb{Q}$ and glue $D^2 \times S^1$ to T_i by a homeomorphism sending $\partial D^2 \times \{y\}$ to a curve representing $\alpha_i[h_i^*] + \beta_i[h]$.

From the Seifert-Van Kampen theorem, when Σ is closed and orientable and $g \geq 0$ then
$$\pi_1(M) = \langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n, h \mid h \text{ central }, \gamma_j^{\alpha_j} = h^{-\beta_j}, [a_1, b_1] \ldots [a_g, b_g]\gamma_1 \ldots \gamma_n = 1\rangle.$$
when nonorientable then $g > 0$ and
$$\pi_1(M) = \langle a_1, \ldots, a_g, \gamma_1, \ldots, \gamma_n, h \mid a_j h a_j^{-1} = h^{-1}, \gamma_j^{\alpha_j} = h^{-\beta_j}, \gamma_j h \gamma_j^{-1} = h, a_1^2 \ldots a_g^2 \gamma_1 \ldots \gamma_n = 1\rangle,$$
One checks that if $\Sigma \neq S^2, \mathbb{R}P^2$ or $\partial\Sigma \neq \emptyset$, then $|H_1(M)| = \infty$ and in these cases left-orderability is dealt with by the Corollary above since the only non-prime SFS is $\mathbb{R}P^3 \# \mathbb{R}P^3$.

It is a fun exercise (for some definition of fun) to show that if $\Sigma = \mathbb{R}P^2$ then the group written above is not left-orderable.

We'll show a simple trick to deal with $H_1(M; \mathbb{Z}) = 0$ and $\Sigma = S^2$.

Theorem 2.34. *If M is a Seifert fibred manifold with $H_1(M; \mathbb{Z}) = 0$ other than the Poincaré homology sphere $\Sigma(2, 3, 5)$ or S^3, then $\pi_1(M)$ is left-orderable.*

Proof. Since we have ruled out S^3 and lens spaces, we arrive at $n \geq 3$. The fundamental group simplifies to be
$$\pi_1(M) = \langle \gamma_1, \ldots, \gamma_n, h \mid h \text{ central }, \gamma_j^{\alpha_j} = h^{-\beta_j}, \gamma_1 \ldots \gamma_n = 1\rangle$$
where the integers $\alpha_i \geq 2$ are pairwise relatively prime (this is a simple exercise using presentation matrices of the abelianization), and we don't have $n = 3$ with $\{\alpha_1, \alpha_2, \alpha_3\} = \{2, 3, 5\}$.

Now consider the group
$$\Delta(\alpha_1, \alpha_2, \alpha_3) = \langle x, y, z \mid x^{\alpha_1} = y^{\alpha_2} = z^{\alpha_3} = xyz = 1\rangle,$$
which is the so-called triangle group. If the $\alpha_i \geq 2$ are pairwise relatively prime, this is an infinite group, with the exception of $\{\alpha_1, \alpha_2, \alpha_3\} = \{2, 3, 5\}$. For all other cases, $\Delta(\alpha_1, \alpha_2, \alpha_3)$ is isomorphic to a group of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2$, a subgroup of index 2 of the group of isometries generated by reflection in the sides of a hyperbolic triangle having angles $\pi/\alpha_1, \pi/\alpha_2, \pi/\alpha_3$.

There is surjective homomorphism $\pi_1(M) \to \Delta(\alpha_1, \alpha_2, \alpha_3)$ given by $\gamma_1 \mapsto x$, $\gamma_2 \mapsto y$, $\gamma_3 \mapsto z$, and all other generators map to the identity. Recalling that $\mathrm{Isom}_+(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$, which is circularly orderable, this defines a map from $\pi_1(M)$ onto an infinite circularly orderable group (it's not too hard to check the group is infinite).

It follows that $\pi_1(M)$ is circularly orderable (since Seifert fibre spaces are irreducible, a fact I have not mentioned yet). Further, since they are irreducible we know that $H^2(\pi_1(M); \mathbb{Z})) \cong H^2(M; \mathbb{Z}) \cong H_1(M; \mathbb{Z}) = 0$, by duality and since M is a $K(\pi, 1)$. But this means that any circular ordering f of $\pi_1(M)$ must satisfy $[f] = id \in H^2(\pi_1(M); \mathbb{Z}))$, so that $\pi_1(M)$ is left-orderable.

□

In fact, the L-space conjecture predicts:

Conjecture 2.35. *If M is a compact, connected, orientable irreducible integer homology 3-sphere other than S^3 or $\Sigma(2,3,5)$ then $\pi_1(M)$ is left-orderable.*

We see from the above argument that, in fact, circular orderability of these groups is equivalent to determining left-orderability.

Also, in regards to left-orderability of the fundamental groups of Seifert fibred manifolds, in fact we can show much more. We let $\widetilde{\mathrm{Homeo}}_+(S^1)$ denote the group of orientation-preserving homeomorphisms of the real line that commute with translation by one.

Theorem 2.36. [7] *Suppose that M is a compact, connected orientable Seifert fibred space. Then if $M \neq S^3$ and $H_1(M;\mathbb{Z})$ is finite then the following are equivalent:*

(1) *$\pi_1(M)$ is left-orderable,*
(2) *$\pi_1(M)$ admits a nontrivial representation $\rho : \pi_1(M) \to \widetilde{\mathrm{Homeo}}_+(S^1)$ with $\rho(h) = sh(1)$, here h is class of the regular fibre and $sh(1)(x) = x + 1$ for all $x \in \mathbb{R}$,*
(3) *M admits a horizontal foliation.*

Determining whether or not these properties hold, in particular (2), is the focus of a series of papers by Eisenbud-Hirsch-Neumann, Jankins-Neumann, and Naimi. It amounts to solving a system of diophantine inequalities involving the α_i's and β_i's.

Lecture 4: Dehn filling and slopes

In this section we will stick to knot complements, but the techniques presented here can be generalized to *knot manifolds*, that is, compact, connected, orientable and irreducible manifolds M other than $S^1 \times S^2$ having boundary a single torus and $H_1(M;\mathbb{Q}) = \mathbb{Q}$.

A *slope* on the boundary of a knot complement M is an element $[\alpha]$ of the projective space $\mathbb{P}H_1(\partial M;\mathbb{R})$ of $H_1(\partial M;\mathbb{R})$ where $\alpha \in H_1(\partial M;\mathbb{R}) \setminus \{0\}$. We set

$$\mathcal{S}(M) = \{\text{slopes on } \partial M\}$$

which we can think of as a copy of S^1.

When $K \subset S^3$ is a (smooth, say) knot, then set $M = S^3 \setminus \nu(K)$, the complement of an open tubular neighbourhood. Then $H_1(\partial M;\mathbb{Z}) \cong \pi_1(\partial M)$ admits a basis $\{\mu, \lambda\}$ where λ is trivial in $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ and μ serves as a generator (i.e. image under inclusion).

With this natural basis we can identify slopes represented by $\alpha \in H_1(M;\mathbb{Z})$ with $\mathbb{Q} \cup \{\infty\}$, by identifying p/q with $\mu^p\lambda^q$ and μ with ∞. We can also identify slopes with ∂M-isotopy classes of essential simple closed curves on ∂M.

Remark 2.37. Using $\mathbb{R}$ coefficients to define slopes is not the usual way things are done, since we often want to focus only on simple closed curves or elements of the fundamental group up to sign. However we will need real coefficients later on.

To each such slope $[\alpha]$ on ∂M we can associate the *α-Dehn filling* of M given by $M(\alpha) = M \cup_f (S^1 \times D^2)$, where $f : \partial(S^1 \times D^2) \to T$ is a homeomorphism for which $f(\{*\} \times \partial D^2)$ is a simple closed curve of slope $[\alpha]$. A standard argument shows that $M(\alpha)$ is independent of the choice of f up to a homeomorphism which is the identity on the complement in M of a collar neighbourhood of T. When $\alpha = \mu^p\lambda^q$ we write $M(p/q)$ in place of $M(\alpha)$. Of course we can generalize this to multiple boundary components, i.e. link complements.

This is a sensible avenue to tackle the L-space conjecture and investigate left-orderability of fundamental groups of all 3-manifolds, since:

Theorem 2.38. *(Lickorish-Wallace) Every 3-manifold arises as a Dehn filling of some link in S^3.*

It's also a good way to produce irreducible M with finite first homology, which is what's left for us to consider after the work of last lecture. The first homology of $M(p/q)$ is $\mathbb{Z}/p\mathbb{Z}$, and the manifold $M(\alpha)$ is reducible for at most three choices of slope α, by Gordon-Luecke.

We can also easily compute the fundamental group of the manifold $M(\alpha)$, e.g. by Seifert-Van Kampen, and it turns out to be

$$\pi_1(M(\alpha)) = \pi_1(M)/\langle\langle \alpha \rangle\rangle;$$

so it is the left-orderability of these quotients that we want to understand.

Owing to the fact that Heegaard Floer homology is well understood with respect to Dehn filling, the L-space conjecture predicts that the following must hold if the conjecture is to be true:

Conjecture 2.39. *Suppose that M is the complement of a knot in S^3. If there exists a rational number $r > 0$ such that $\pi_1(M(r))$ is not left-orderable, then $\pi_1(M(p/q))$ is left-orderable if and only if $p/q \in (-\infty, 2g(K) - 1)$.*

Remark 2.40. Two comments are in order:

(1) Here, $g(K)$ is the knot genus, i.e. the minimal genus of a Seifert surface for the knot K.
(2) By $-\infty$ we mean the fraction $1/0$, corresponding to a simple closed curve homotopic to μ. Then $M(\mu) \cong S^3$, and our convention for this lecture is that the trivial group *is not* left-orderable.

The takeaway here is that sometimes these manifolds have left-orderable fundamental group, other times not. So we need techniques to order these groups, and to obstruct the existence of orderings of these groups. We present the basics of both below, indicating directions for possible generalizations (and research!).

Techniques for left-ordering Dehn fillings. Here, the general scheme of many techniques is as follows:

(1) Find a technique for creating representations $\rho_s : \pi_1(M) \to \mathrm{Homeo}_+(S^1)$ with non-cyclic image.
(2) Given $\alpha = \mu^p\lambda^q$, find s such that the representation ρ_s factors through the quotient $\pi_1(M)/\langle\langle \alpha \rangle\rangle$.
(3) Use these representations to create circular orderings f of $\pi_1(M(\alpha))$, and adjust the ordering so that $[f] = id \in H_1(M(\alpha); \mathbb{Z})$.

In this section we'll deal with a concrete example following Boyer-Rolfsen-Wiest [7], but indicate possible generalizations afterwards. We begin with a lemma:

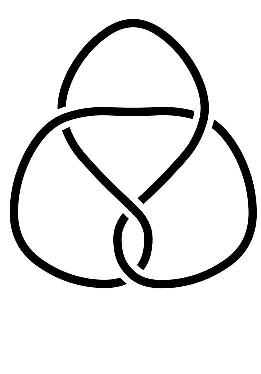
Lemma 2.41. *Every nonabelian circularly orderable group is infinite.*

Proof. If f is a circular ordering of a finite group G, then we arrive at a central extension

$$0 \to \mathbb{Z} \to \widetilde{G}_f \to G \to 1$$

where $\widetilde{G}_f$ is left-orderable. Then $\widetilde{G}_f$ is torsion-free, and has an infinite cyclic subgroup of finite index, which implies that $\widetilde{G}_f$ itself is infinite cyclic (In fact, we don't need to do something so tricky here–we can just remark that the lifted ordering on $\widetilde{G}_f$ is Archimedean, so it is an abelian group. But now it's a finitely generated torsion free abelian group, so cyclic). But then the quotient G is also cyclic, so finite circularly orderable groups are cyclic–hence abelian. $\square$

With this fact for future use, we will begin with our favourite knot:



We calculate that the knot group is

$$G = \langle x, y \mid wx = yw \rangle$$

where $w = xy^{-1}x^{-1}y$. We can observe that $\mu = x$ serves as a meridian, and $\lambda = yx^{-1}y^{-1}x^2y^{-1}x^{-1}y$ serves as a longitude. In particular, $[\mu, \lambda] = id$.

Now it happens that the representations with nonabelian image $\rho : G \to \mathrm{PSL}(2, \mathbb{C})$ are completely described the so-called Riley polynomial [28], and by finding real solutions to this polynomial we can discover representations $\rho : G \to \mathrm{PSL}(2, \mathbb{R})$. For this particular knot, we can go through the exercise and find that for $s \geq \frac{1+\sqrt{5}}{2}$ and for

$$t = \frac{1 + \sqrt{(s - s^{-1})^4 + 2(s - s^{-1})^2 - 3}}{2(s - s^{-1})} \in \mathbb{R},$$

we get a representation $\rho_s : G \to \mathrm{PSL}(2, \mathbb{R})$ by setting

$$\rho_s(x) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \text{ and } \rho_s(y) = \begin{pmatrix} \frac{s+s^{-1}}{2} + t & \frac{s-s^{-1}}{2} + t \\ \frac{s-s^{-1}}{2} - t & \frac{s+s^{-1}}{2} - t \end{pmatrix}.$$

Next, we can observe that the image of every ρ_s is nonabelian, since (for example) $\rho_s(x)$ and $\rho_s(y)$ never commute since they are not both diagonal. Since $\mathrm{PSL}(2, \mathbb{R})$ is circularly orderable, by our lemma this observation implies that the image of ρ is always an infinite circularly orderable group.

Now we want to find all $p/q \in \mathbb{Q}$ for which there exists $s \in \mathbb{R}$ such that $\rho_s(\mu^p\lambda^q) = \pm I$, the identity matrix. We can solve directly, first we observe that since $\rho_s(x)$ is diagonal and not the identity, $\rho_s(\lambda)$ must also be diagonal since it commutes with $\rho_s(x)$.

Whatever the matrix $\rho_s(\lambda)$ is, all we need to do to guarantee $\rho(\mu^p\lambda^q) = \pm I$ is confirm that the $(1, 1)$ entry of $\rho(\mu^p\lambda^q)$ is equal to one. So while the $(1, 1)$ entry of $\rho(\mu)$ is s, we'll write $\psi(s)$ for the $(1, 1)$ entry of $\rho(\lambda)$, whatever it happens to be.

Then $\rho(\mu^p\lambda^q) = \pm I$ if and only if $s^p\psi(s)^q = \pm 1$, or

$$-\frac{ln|\psi(s)|}{ln|s|} = \frac{p}{q},$$

and so we have reduced the problem to determining the range of the function

$$g(s) = -\frac{ln|\psi(s)|}{ln|s|}, \text{ where } s \geq \frac{1+\sqrt{5}}{2}.$$

At this point, there's essentially a first-year calculus type of computation that shows $[0, 4) \subset g([\frac{1+\sqrt{5}}{2}, \infty))$. Therefore for all $p/q \in [0, 4)$, there exists a representation $\rho_s : G \to \mathrm{PSL}(2, \mathbb{R})$ with infinite image and satisying $\rho_s(\mu^p\lambda^q) = \pm I$. We conclude that for $p/q \in [0, 4)$, there are representations

$$\pi_1(M(p/q)) \to \mathrm{PSL}(2, \mathbb{R})$$

with infinite image, and so $\pi_1(M(p/q))$ is circularly orderable. But then because $H_1(\pi_1(M(p/q)); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, for every such group there is a finite index normal left-orderable subgroup $H \subset \pi_1(M(p/q))$ with H left-orderable, with index dividing p.

Remark 2.42. Some remarks on improving this result:

(1) Each representation $\rho_s : \pi_1(M(p/q)) \to \mathrm{PSL}(2, \mathbb{R})$ yields a circular ordering f of $\pi_1(M(p/q))$ about which we know almost nothing (we haven't tried!). With some care, we can improve the construction above to yield circular orderings with $[f] = id \in H^2(\pi_1(M(p/q); \mathbb{Z})$ so that $\pi_1(M(p/q)$ is left-orderable for all $p/q \in [0, 4)$ [6].

(2) The figure eight knot is special in that there's an automorphism $\psi : G \to G$ that satisfies $\psi(\mu) = \mu$ and $\psi(\lambda) = -\lambda$. Therefore every representation $\rho_s : \pi_1(M(p/q)) \to \mathrm{PSL}(2,\mathbb{R})$ yields a representation $\rho_s' : \pi_1(M(-p/q)) \to \mathrm{PSL}(2,\mathbb{R})$ as well, so $\pi_1(M(p/q)$ is left-orderable for all $p/q \in (-4, 4)$.
(3) This technique—using the Riley polynomial—has been pushed to its limit in many, many papers. Each produces a range of left-orderable fillings for various families of 2-bridge knots.
(4) There is nothing special about representations $\pi_1(M(p/q)) \to \mathrm{PSL}(2,\mathbb{R})$, aside from having circularly orderable image. Other techniques for producing such representations will work as well, namely foliations and Anosov flows combined with universal circle constructions of Thurston/Fenley ([11, 20]) can produce families of representations
$$\pi_1(M(p/q)) \to \mathrm{Homeo}_+(S^1),$$
and then Euler class type arguments in order to produce lifts will similarly produce ranges of left-orderable Dehn fillings.

Techniques for obstructing left-orderings of Dehn filled manifolds.

Here we need a new object. For a left-orderable group G, set
$$\mathrm{LO}(G) = \{P \subset G \mid \ P \text{ is the positive cone of a left-ordering}\}.$$
Note that this is a subset of the power set $\mathcal{P}(G)$, which we can identify with $\{0,1\}^G$. We can equip $\{0,1\}$ with the discrete topology, and $\{0,1\}^G$ with the product topology, and then give $\mathrm{LO}(G) \subset \mathcal{P}(G)$ the subspace topology.

A more concrete description is: We give $\mathrm{LO}(G)$ the topology generated by the subbasic open sets $U_g = \{P \in \mathrm{LO}(G) \mid g \in P\}$ for all $g \in G \setminus \{id\}$, so arbitrary open sets look like
$$\bigcap_{i=1}^{n} U_{g_i} = \{\text{positive cones containing } g_1, \ldots, g_n\} = \{\text{orderings where } g_1, \ldots, g_n \text{ are positive}\}.$$
This space of left-orderings of G is a compact, totally disconnected space, and if G is countable, then it is metrizable. Whenever $H \leq G$, we can define the restriction map $r : \mathrm{LO}(G) \to \mathrm{LO}(H)$, given by $r(P) = P \cap H$. This map happens to be continuous.

The space $\mathrm{LO}(\mathbb{Z}^2)$**.** Given a line $L \subset \mathbb{R}^2$, there are two orderings of $\mathbb{Z}^2$ determined by L if the slope is irrational, and four if rational. When the line L has irrational slope, the positive cones of the two corresponding orderings are given by declaring all elements to a given side of L to be positive. When L has rational slope, then $L \cap \mathbb{Z}^2 \cong \mathbb{Z}$, and we make four lexicographic orderings of $\mathbb{Z}^2$ from the short exact sequence
$$0 \to L \cap \mathbb{Z}^2 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.$$
Conversely, one can show that every ordering of $\mathbb{Z}^2$ with positive cone P uniquely determines a line as follows. First, uniquely extend the positive cone P of $\mathbb{Z}^2$ to positive cone P' of $\mathbb{Q}^2$ by declaring $(p_1/q_1, p_2/q_2) > 0$ if and only if $(p_1q_2, p_2q_1) \in P$ (i.e. we clear denominators by multiplying by $q = q_1q_2$). Set
$$L(P) = \{x \in \mathbb{R}^2 \mid \text{ every nbhd of } x \text{ contains elements of } P' \text{ and } (P')^{-1} \}.$$
One can check that indeed, L is a line.

Thus there is a map $\mathrm{LO}(\mathbb{Z}^2) \to \mathbb{R}P^1$ given by $P \mapsto [L(P)]$, in particular there's a map $\mathrm{LO}(H_1(\partial M;\mathbb{Z})) \to \mathcal{S}(M)$ by sending a positive cone to its corresponding slope. Define the *slope map* to be the composition
$$s : \mathrm{LO}(\pi_1(M)) \xrightarrow{r} \mathrm{LO}(\pi_1(\partial M)) = \mathrm{LO}(H_1(\partial M;\mathbb{Z})) \to \mathcal{S}(M),$$
defined by $s(P) = [L(P \cap \pi_1(\partial M))]$. In plain terms:

We start with an ordering of $\pi_1(M)$, and restrict it to the subgroup $\pi_1(\partial M)$, which is just a copy of $\mathbb{Z}^2$ since inclusion of the torus boundary induces an injective map on the level of fundamental groups. Then look at the line determined by this ordering.

Proposition 2.43 ([16]). *Suppose that M is the complement of a knot in S^3. If $M(\alpha)$ is LO then $[\alpha]$ is in the image of the slope map.*

Proof. Suppose that $L \subset \mathbb{R}^2$ is the line corresponding to the slope $[\alpha]$, i.e. $[L] = [\alpha]$. To prove the proposition, we must show that there exists an ordering of $\pi_1(\partial M)$ that:

(1) Is defined lexicographically from the short exact sequence

$$0 \to L \cap \pi_1(\partial M) \to \pi_1(\partial M) \to \mathbb{Z} \to 1,$$

so that the order maps to the correct slope, and

(2) is in the image of the restriction map $r : \mathrm{LO}(\pi_1(M)) \to \mathrm{LO}(\pi_1(\partial M))$.

To this end, we consider the short exact sequence

$$1 \to \langle\langle \alpha \rangle\rangle \xrightarrow{i} \pi_1(M) \xrightarrow{q} \pi_1(M(\alpha)) \to 1.$$

By assumption $\pi_1(M(\alpha))$ is left-orderable, and since this means the normal closure $\langle\langle \alpha \rangle\rangle$ is infinite index in $\pi_1(M)$, it's left-orderable, too. Choose positive cones $Q \subset \langle\langle \alpha \rangle\rangle$ and $R \subset \pi_1(M(\alpha))$ and set $P = i(Q) \cup q^{-1}(R)$.

Then we observe that $\langle\langle \alpha \rangle\rangle \cap \pi_1(\partial M) = \langle \alpha \rangle$, because if it were rank two then the quotient would be not left-orderable (either trivial or contain torsion). Therefore the restriction $r(P) = P \cap \pi_1(\partial M)$ defines an ordering of $\pi_1(\partial M)$ that is lexicographic as in (1) above, with (2) being automatic. Therefore $s(P) = [\alpha]$. □

Now we make a simple observation, again restricting to knots in S^3 to keep things simple:

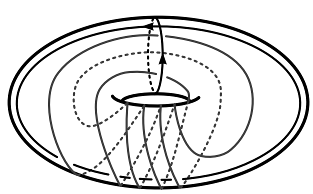
Proposition 2.44 ([16]). *Suppose that we have integers $p, q, r, s > 0$ and that $p/q > r/s > 0$, and that M is the complement of a knot in S^3 with meridian and longitude $\{\mu\lambda\}$. If every positive cone $P \in \pi_1(M)$ satisfies:*

$$\mu^p \lambda^q \in P \Rightarrow \mu^r \lambda^s \in P,$$

then for every $a/b \in (r/s, p/q)$ the group $\pi_1(M(a/b))$ is not left-orderable.

Proof. If $\pi_1(M(a/b))$ were left-orderable, we would be able to construct a positive cone $P \subset \pi_1(M)$ satisfying $s(P) = a/b$. However any such positive cone would yield an ordering of $\pi_1(M)$ where $\mu^p\lambda^q$ and $\mu^r\lambda^s$ have opposite signs, since they lie on opposite sides of the line $L(P) \subset \mathbb{R}^2$. □

Example 2.45. Consider a (p, q) torus knot in S^3 and its complement $M_{p,q}$. For instance the $(3, 5)$ torus knot is this:



The knot group in this case is $\pi_1(M_{p,q}) = G_{p,q} = \langle a, b \mid a^p = b^q \rangle$. The meridian and longitude are $\mu = b^j a^i$, where i, j are integers satisfying $pj + qi = 1$, we may assume that $p > i > 0$ and $0 > j > -q$. The longitude is given by $\lambda = \mu^{-pq} a^p$.

Now observe that the a^p corresponds to the slope $\mu^{pq}\lambda$, i.e. $\frac{pq}{1} = pq$. Assume that we have a left-ordering of $G_{p,q}$ with $\mu^{pq}\lambda = a^p > id$.

Then $a^p = b^q > id$, so in fact both $a, b > id$ as well. But then $\mu^{pq-1}\lambda = \mu^{pq}\lambda\mu^{-1} = a^p a^{-i} b^{-j} = a^{p-i} b^{-j}$ is a product of positive elements so $\mu^{pq-1}\lambda > id$. Similarly $\mu^{pq+1}\lambda = \mu^{pq}\lambda\mu = b^q(b^j a^i) = b^{q+j} a^i > id$, as it is a product of positive elements.

This shows that no slope in the interval $[pq-1, pq+1]$ gives a left-orderable group upon Dehn filling the complement of the torus knot.

In fact, this computation can be improved because the groups $G_{p,q}$ are one of the few cases where we know exactly the image of the slope map. One can show:

$$s(\mathrm{LO}(G_{pq,})) = [-\infty, pq-p-q],$$

so potentially all the slopes in that interval yield left-orderable fundamental groups upon Dehn filling. □

Example 2.46. The converse of Proposition 2.44 does not hold, we can provide an explicit positive cone P of $G_{3,2} = \pi_1(M_{3,2})$ such that $s(P) = [\mu]$, but filling along μ gives S^3 and so the result is *not* a left-orderable fundamental group. Here is the ordering:

First, we note that

$$G_{2,3} = \langle a, b \mid a^3 = b^2 \rangle$$

is isomorphic to the three-strand braid group

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle,$$

the isomorphism is given by $a \mapsto \sigma_1\sigma_2$ and $b \mapsto \sigma_1\sigma_2\sigma_1$. The meridian $\mu = b^{-1}a^2$ becomes σ_2 in our new generators.

Define a positive cone $P_D \subset B_3$ as follows.

A word w in the generators of B_3 will be called 1-*positive* if all occurences of σ_1 in w have positive exponent. An element $\beta \in B_3$ will be called 1-positive if it admits a 1-positive representative word, and we set

$$P_1 = \{\beta \in B_3 \mid \beta \text{ is 1-positive}\}.$$

Now we define

$$P_D = \{\sigma_2^k\}_{k>0} \cup P_1,$$

and we have a theorem:

Theorem 2.47. *(Dehornoy,* [17]*) The set P_D is the positive cone of a left-ordering of B_3, called the* Dehornoy ordering.

E.g. the braid $\sigma_1\sigma_2^{-1}$ is positive, since it is written as a word having only positive occurences of σ_1. The sign of the element $\sigma_1^{-1}\sigma_2\sigma_1$ is not immediately clear, but we can rewrite it as

$$\sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^{-1}$$

and observe that since the new word $\sigma_2\sigma_1\sigma_2^{-1}$ has σ_1 with only positive powers, it's positive. From the definition, it is clear that P_D is a semigroup, but Dehornoy's main contribution is that $P_D \cup P_D^{-1} = B_3 \setminus \{id\}$ and $P_D \cap P_D^{-1} = \emptyset$.

Lemma 2.48. *The braid σ_2 is the least element of P_D.*

Proof. Suppose that $\beta \in P_D$, let $<_D$ denote the corresponding left-ordering. If $\beta = \sigma_2^k$ for some $k > 1$, then $\sigma_2 <_D \beta$. On the other hand if $\beta \in P_1$, then we can choose a representative word w having only positive occurences of σ_1. But then $\sigma_2^{-1}w$ also has only positive occurences of σ_1, meaning $\sigma_2^{-1}\beta \in P_D$, i.e. $\sigma_2^{-1} <_D \beta$. □

Because of this, we know that $\sigma_2 = \mu$ must be the smallest element of $\pi_1(\partial M)$ in the restriction ordering $P_D \cap \pi_1(\partial M)$. This is only possible if the restriction ordering is lexicographic relative to the short exact sequence

$$0 \to \langle \mu \rangle \to \pi_1(\partial M) \to \mathbb{Z} \to 0,$$

i.e. $s(P_D) = [\mu]$.

A conjectural picture: Determining the image of this map is related to ongoing work of myself, Steve Boyer, Ying Hu, Cameron Gordon ([5, 4]), and implicitly many others (they aren't necessarily using this language in their work). Again, I state the picture just when M is the complement of a knot in S^3.

Either:

(1) $s(\mathrm{LO}(\pi_1(M))) = \mathcal{S}(M)$, i.e. the slope map is surjective, and every filling $\pi_1(M(\alpha))$ is left-orderable except for $\alpha = \mu$, or
(2) $s(\mathrm{LO}(\pi_1(M))) = [-\infty, 2g(K)-1]$ and $\pi_1(M(p/q))$ is left-orderable for all $p/q \in (-\infty, 2g(K)-1)$.

This is currently wide open, as being in the image of the slope map is, a priori, very different than having a left-orderable quotient.

Lecture 5: Amalgams and JSJ decompositions

We already saw how to left-ordering free products, and used this to reduce the question of left-ordering fundamental groups of 3-manifolds to the situation where M is irreducible. There is another canonical decomposition in 3-manifold theory, namely the JSJ decomposition, which allows one to express the fundamental group as a certain kind of free product with amalgamation. So we investigate left-orderability of these groups, with an eye towards using the JSJ decomposition.

If M is orientable, irreducible and closed, the JSJ decomposition of M provides a unique (up to isotopy) minimal collection $\mathcal{T}$ of embedded, disjoint incompressible tori such that $M \setminus \mathcal{T}$ consists of pieces $M_1, \ldots, M_n$ where each M_i is either Seifert fibered or atoroidal. This decomposition allows one to realise $\pi_1(M)$ as the fundamental group of a graph of groups whose vertex groups are $\pi_1(M_1), \ldots, \pi_1(M_n)$ and whose edge groups are $\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$, where T ranges over all tori in the collection $\mathcal{T}$.

By our previous lectures, if M contains essential tori, then $\pi_1(M)$ is thus expressible as a fundamental group of a graph of groups, all of whose edge groups and vertex groups are left-orderable. As no obstruction to left-orderability arises from considering these groups independently, the key to understanding left-orderability of $\pi_1(M)$ therefore lies in an analysis of the gluing maps used to reassemble M from the pieces M_i, and the behaviour of the left-orderings of each $\pi_1(M_i)$ restricted to the components of ∂M_i with respect to these gluing maps. So, for example, in the case of two pieces we have two fundamental groups of 3-manifolds with torus boundary amalgamated along a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Unfortunately the situation is not so easy when it comes to free products with amalgamation. Suppose that A, G, H are groups equipped with injective homomorphisms $\phi_1 : A \to G$, $\phi_2 : A \to H$, and let $S \subset G * H$ denote the set

$$S = \{\phi_1(a)\phi_2(a^{-1}) \mid a \in A\}.$$

The free product of G and H amalgamated along the ϕ_i's is the quotient group

$$G *_{\phi_i} H = G * H \big/ \langle\langle S \rangle\rangle .$$

This group is not always left-orderable, as the following example shows, while our experience with free products tells us that sometimes (e.g. for trivial amalgamations) it will certainly be a LO group.

Example 2.49. We construct the twisted I-bundle over the Klein bottle as follows. Consider $\mathbb{R}^2 \times [-1/2, 1/2]$ and the action on this space by the group K generated by the maps

$$f(x, y, z) = (x + 1, y, z), \text{ and } g(x, y, z) = (-x, y + 1, -z).$$

Set $N = (\mathbb{R}^2 \times [-1/2, 1/2])/K$, one can check that this defines a Seifert fibred 3-manifold with torus boundary. Observing that $fgf^{-1} = g^{-1}$, its fundamental group is

$$K = \langle x, y \mid xyx^{-1} = y^{-1} \rangle,$$

and $\pi_1(\partial N) = \langle x^2, y \rangle$. This group is left-orderable, in fact with only four left-orderings coming lexicographically from

$$1 \to \langle y \rangle \to K \to \mathbb{Z} \to 0.$$

Set $K_i = \langle x_i, y_i \mid x_i y_i x_i^{-1} = y_i^{-1} \rangle$ for $i = 1, 2$. Let $A = \mathbb{Z} \oplus \mathbb{Z}$. Note that for each i, the subgroup $\langle y_i, x_i^2 \rangle$ is isomorphic to A. Define $\phi_1 : A \to K_1$ by $\phi_1(0, 1) = y_1$, and $\phi_1(1, 0) = x_1^2$ while $\phi_2 : A \to K_1$ is given by $\phi_2(0, 1) = x_2^2$, and $\phi_2(1, 0) = y_2$.

Next observe that both groups K_i are left-orderable since they fit into a short exact sequence with infinite cyclic kernel and quotient. Moreover, in every left-ordering of K_i with $id < y_i$ (there is at least one of these) we must have $y_i < x_i$ and therefore $y_i < x_i^2$. To see this, suppose not, say $x_i < y_i$. Then $y_i^{-1} x_i < id$, and since $x_i^{-1} < id$, so we also have $x_i^{-1} y_i^{-1} x_i < id$. But then $x_i^{-1} y_i^{-1} x_i = y_i < id$ because $x_i y_i x_i^{-1} = y_i^{-1}$, this is a contradiction.

Now considering the free product with amalgamation $K_1 *_{\phi_i} K_2$, suppose that it is left-orderable. Then the argument above applied to $K_1 \subset K_1 *_{\phi_i} K_2$ tells us that we must have $y_1 < x_1^2$ in every left-ordering of $K_1 *_{\phi_i} K_2$. On the other hand, $x_1^2 = \phi_1(1, 0) = \phi_2(1, 0) = y_2$ and $y_1 = \phi_1(0, 1) = \phi_2(0, 1) = x_2^2$, so this inequality forces $x_2^2 < y_2$, which is not possible. So $K_1 *_{\phi_i} K_2$ must not be left-orderable.

Note that $K_1 *_{\phi_i} K_2$ is the fundamental group of a 3-manifold $W = N_1 \cup_\psi N_2$ where N_i are copies of the twisted I-bundle over the Klein bottle, and $\psi : \partial N_1 \to \partial N_2$ is a homeomorphism between their boundary tori inducing the homomorphism $\phi_2 \circ \phi_1^{-1} : \pi_1(\partial N_1) \to A \to \pi_1(\partial N_2)$. □

So, not only do we sometimes get non-LO groups, we even get non-LO groups in the case where we're working with fundamental groups of 3-manifolds glued along incompressible torus boundary components.

However, we do know necessary and sufficient conditions. First, some notation. For a LO group G, recall

$$\mathrm{LO}(G) = \{P \subset G \mid P \text{ is a positive cone }\}.$$

Note that $\mathrm{LO}(G)$ has an action by conjugation, because when P is a positive cone, so is gPg^{-1} (it happens this is an action by homeomorphisms). A family $N \subset \mathrm{LO}(G)$ is called *normal* if it is invariant under this G-action, i.e. $P \in N \Rightarrow gPg^{-1} \in N$ for all $g \in G$. There is a much more general statement of the following theorem that holds for general amalgams and fundamental groups of graphs of groups (due to Chiswell, [12]), but we will stick to the case of two factors.

Theorem 2.50 (Bludov-Glass [3])**.** *Suppose that A, G, H are groups equipped with injective homomorphisms $\phi_1 : A \to G$, $\phi_2 : A \to H$. The free product with amalgamation $G *_{\phi_i} H$ is left-orderable if and only if there exist normal families $N_1 \subset \mathrm{LO}(G)$ and $N_2 \subset \mathrm{LO}(H)$ satisfying*

$$(\forall P \in N_i)(\exists Q \in N_j) \text{ such that } \phi_i^{-1}(P) = \phi_j^{-1}(Q)$$

whenever $i, j \in \{1, 2\}$.

The proof is well beyond the scope of these notes. The basic idea is to use the normal families, together with some sophisticated set-theoretic constructions, to create a totally ordered set $(X, <)$ which admits an effective, order-preserving action by $G *_{\phi_i} H$.

Despite the rather technical conditions, this theorem already means that certain types of free products with amalgamation are always LO.

Corollary 2.51. *Suppose that G, H, A are as above, that G and H are LO and A is infinite cyclic. Then $G *_{\phi_i} H$ is LO.*

Proof. Just take $N_1 = \mathrm{LO}(G)$ and $N_2 = \mathrm{LO}(H)$. These are certainly normal, and if $P \in N_1$ then there are only two possibilities for $\phi_1^{-1}(P)$ since $\mathbb{Z} \cong A$ has only two left-orderings, i.e. we conclude

$$\mathrm{LO}(A) = \{\phi_i^{-1}(P) \mid P \in N_i\}$$

for $i = 1, 2$. This means that for $P \in N_1$ there is always a $Q \in N_2$ such that $\phi_2^{-1}(Q) = \phi_1^{-1}(P)$.

This situation is entirely symmetric so the same argument shows every $\phi_2^{-1}(Q)$ for $Q \in N_2$ has a corresponding $P \in N_1$. □

This actually generalizes in many ways to larger classes of groups, e.g.

Theorem 2.52. *Suppose that G, H, A are as above, that G and H are LO and nilpotent. Then $G *_{\phi_i} H$ is left-orderable.*

Proof. We'll use one black box here, which is a result due to E. Formanek [21]:

> If $P \subset G \setminus \{id\}$ is a semigroup and G is nilpotent, then there exists a positive cone $Q \subset G$ with $P \subset Q$.

So we can apply the same argument as in the previous proposition, setting $N_1 = \mathrm{LO}(G)$ and $N_2 = \mathrm{LO}(H)$ since $\mathrm{LO}(A) = \{\phi_i^{-1}(P) \mid P \in N_i\}$. □

The case of amalgamation along a cyclic group is already enormously useful in the case of 3-manfiolds having incompressible torus boundary. In the proof below we follow [14].

Theorem 2.53. [14] *Suppose that M_1 and M_2 are 3-manifolds with incompressible torus boundaries, and $\phi : \partial M_1 \to \partial M_2$ is a homeomorphism such that $W = M_1 \cup_\phi M_2$ is irreducible. If there exists a slope α such that $\pi_1(M_1(\alpha))$ and $\pi_1(M_2(\phi_*(\alpha)))$ are both left-orderable, then $\pi_1(W)$ is also left-orderable (here, ϕ_* is the induced homomorphism on fundamental groups).*

Proof. Let G_i denote the fundamental group $\pi_1(M_i)$ for $i = 1, 2$, each equipped with an inclusion $f_i : \mathbb{Z} \oplus \mathbb{Z} \to G_i$ that identifies the peripheral subgroup $\pi_1(\partial M_i)$ with $\mathbb{Z} \oplus \mathbb{Z}$, satisfying $\phi_* \circ f_1 = f_2$. Write $q_1 : G_1 \to G_1/\langle\langle \alpha \rangle\rangle$ and $q_2 : G_2 \to G_2/\langle\langle \phi_*(\alpha) \rangle\rangle$ for the natural quotient maps.

Suppose that $\pi_1(M(\alpha))$ and $\pi_1(M_2(\phi_*(\alpha)))$ are both left-orderable, and consider $\langle\langle \alpha \rangle\rangle \cap \pi_1(\partial M_1)$. Since this intersection is a nontrivial subgroup of $\pi_1(\partial M_1) \cong \mathbb{Z} \oplus \mathbb{Z}$ and α is primitive, the intersection is isomorphic to either $\mathbb{Z} \cong \langle \alpha \rangle, \mathbb{Z} \oplus n\mathbb{Z} \subset \pi_1(\partial M_1)$, or $\mathbb{Z} \oplus \mathbb{Z} \cong \pi_1(\partial M_1)$. If $\langle\langle \alpha \rangle\rangle \cap \pi_1(\partial M_1) \cong \mathbb{Z} \oplus n\mathbb{Z}$, then the quotient $\pi_1(M(\alpha))$ would have torsion, so this case does not arise when $\pi_1(M_1(\alpha))$ is left-orderable, the same holds for $\pi_1(M_2(\phi_*(\alpha)))$, so we break our proof into two cases.

First consider the case where $\langle\langle \alpha \rangle\rangle \cap \pi_1(\partial M_1)$ and $\langle\langle \phi_*(\alpha) \rangle\rangle \cap \pi_1(\partial M_2)$ are both infinite cyclic. Here, ϕ induces an isomorphism $\bar{\phi}$ between subgroups

$$\bar{\phi} : q_1(\pi_1(\partial M_1)) \to q_2(\pi_1(\partial M_2)),$$

satisfying $\bar{\phi} \circ q_1 \circ f_1 = q_2 \circ f_2$. The subgroups $q_1(\pi_1(\partial M_1))$ and $q_2(\pi_1(\partial M_2))$ are both infinite cyclic, and by the universal property for pushouts, we have a unique homomorphism

$$h : G_1 *_\phi G_2 \longrightarrow G_1/\langle\langle \alpha \rangle\rangle *_{\bar{\phi}} G_2/\langle\langle \phi_*(\alpha) \rangle\rangle$$

resulting from the following diagram:

$$\begin{tikzcd}
\mathbb{Z} \oplus \mathbb{Z} \arrow[r, "f_2"] \arrow[d, "f_1"'] & G_2 \arrow[d] \arrow[rr, bend left, "q_2"] & & \\
G_1 \arrow[r] \arrow[dr, bend right, "q_1"'] & G_1 *_\phi G_2 \arrow[drr, dashed, "h"] & & G_2/\langle\langle \phi_*(\alpha) \rangle\rangle \arrow[d] \\
 & G_1/\langle\langle \alpha \rangle\rangle \arrow[rr] & & G_1/\langle\langle \alpha \rangle\rangle *_{\bar{\phi}} G_2/\langle\langle \phi_*(\alpha) \rangle\rangle
\end{tikzcd}$$

.

Note that h is nontrivial (in fact, surjective) since the maps q_1 and q_2 are surjective.

Because the groups $\pi_1(M_1(\alpha))$ and $\pi_1(M_2(\phi_*(\alpha)))$ are both left-orderable, the group $G_1/\langle\langle\alpha\rangle\rangle *_{\overline{\phi}} G_2/\langle\langle\phi_*(\alpha)\rangle\rangle$ is a free product of left-orderable groups amalgamated along a cyclic subgroup. The image of the map h is therefore a left-orderable group by the previous result, so that $\pi_1(W) \cong G_1 *_\phi G_2$ is left-orderable.

On the other hand, suppose that either $\langle\langle\alpha\rangle\rangle \cap \pi_1(\partial M_1) = \pi_1(\partial M_1)$, or $\langle\langle\phi_*(\alpha)\rangle\rangle \cap \pi_1(\partial M_2) = \pi_1(\partial M_2)$, or both. Without loss of generality, suppose that α satisfies $\langle\langle\alpha\rangle\rangle \cap \pi_1(\partial M_1) = \pi_1(\partial M_1)$. In this setting we have an alternative construction for h as follows.

$$\begin{tikzcd}
\mathbb{Z} \oplus \mathbb{Z} \arrow[r, "f_2"] \arrow[d, "f_1"'] & G_2 \arrow[d] \arrow[ddr, bend left, "1"] & \\
G_1 \arrow[r] \arrow[drr, bend right, "q_1"'] & G_1 *_\phi G_2 \arrow[dr, dashed, "h"] & \\
& & G_1/\langle\langle\alpha\rangle\rangle
\end{tikzcd}$$

Note that this is well defined since $\langle\langle\alpha\rangle\rangle$ contains the entire peripheral subgroup $\pi_1(\partial M_1)$, and h is again surjective. Now as $\pi_1(M_1(\alpha))$ is left-orderable, $G_1/\langle\langle\alpha\rangle\rangle$ is left-orderable and h provides the required homomorphism to a left-orderable group so that $\pi_1(W)$ is left-orderable. $\square$

This technique is powerful, in the sense that we can do things like left-order all fundamental groups of integer homology sphere graph manifolds.

Theorem 2.54. [14] *Suppose M is an irreducible, toroidal graph manifold (i.e. it is made by gluing together SF pieces). If M is an integer homology 3-sphere, then $\pi_1(M)$ is LO.*

But does not handle all kinds of 3-manifolds W that can arise from gluing together manifolds along incompressible torus boundary components. Here is a good example of a manifold for which this trick won't work.

Example 2.55. [14] Here are the two pieces we will glue together. Our first piece, M_1, will be the complement of the trefoil in S^3. Recall from the last lecture that we noted $\pi_1(M_1)$ is isomorphic to the three-strand braid group

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle,$$

and that $\mu = \sigma_2$. We can also compute that $\lambda = \Delta^2\sigma_2^{-6}$, where

$$\Delta = \sigma_1\sigma_2\sigma_1.$$

Our second piece M_2 will be a copy of the twisted I-bundle over the Klein bottle, so that $\pi_1(M_2)$ is the group

$$\langle x, y | xyx^{-1} = y^{-1} \rangle,$$

and $\pi_1(\partial M_2) = \langle y, x^2 \rangle$.

Now define

$$W := M_1 \cup_\phi M_2$$

where ϕ is the gluing map of their boundaries defined on the peripheral subgroups by the formula

$$\phi(\sigma_2) = y^{-1}, \phi(\Delta^2) = y^{-1}x^2.$$

By applying the Seifert-Van Kampen theorem, we see that

$$\pi_1(W) = \langle \sigma_1, \sigma_2, x, y | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, xyx^{-1} = y^{-1}, \sigma_2 = y^{-1}, \Delta^2 = y^{-1}x^2 \rangle.$$

It is not hard to check that $\pi_1(W)$ abelianizes to give $\mathbb{Z}/4\mathbb{Z}$, so W is not an integer homology sphere. In fact, W arises from $+4$-surgery on the figure eight knot.

Proposition 2.56. *The fundamental group of W cannot be left-ordered by applying the previous theorem.*

Proof. In order to apply the previous result, we must find $\alpha \in \mathcal{S}(M_1)$ such that $\pi_1(M_1(\alpha))$ and $\pi_1(M_2(\phi_*(\alpha)))$ are left-orderable. However, one can check that for $\alpha \in \langle y, x^2 \rangle$ the group

$$\pi_1(M_2)/\langle\langle \alpha \rangle\rangle = \pi_1(M_2(\alpha))$$

is left-orderable if and only if $\alpha = y$, in fact if $\alpha \neq y$ then the resulting group always has torsion (indeed, is a finite group if $\alpha \notin \langle y \rangle$.)

Therefore, if we are to the previous theorem to W, we must take $\phi(\alpha) = y$ to be our left-orderable slope on ∂M_2. Correspondingly, we must have $\alpha = \phi^{-1}(y) = \sigma_2 = \mu$. However, $\mu = \sigma_2$ is not a left-orderable slope since $B_3/\langle\langle \sigma_2 \rangle\rangle$ is trivial. □

This shows we actually need the full generality of the theorem, and so we *must* deal with normal families and the full structure of $\mathrm{LO}(G)$.

Example 2.57. Continuing with the last example: Recall from last class that we defined the Dehornoy ordering of B_3 to have positive cone

$$P_D = \{\sigma_2^k\}_{k>0} \cup P_1,$$

where P_1 was the 1-positive elements of B_3. From the basic properties of P_D, we can argue that

$$P_D^{\pm\pm} = \{\sigma_2^{\pm k}\}_{k>0} \cup P_1^{\pm}$$

actually defines a positive cone in B_3 for all choices of $\pm$. So we get four positive cones, with $P_D = P_D^{++}$ being the original. Set

$$N_1 = \text{ normal closure of } \{P_D^{++}, P_D^{+-}, P_D^{-+}, P_D^{--}\} \subset \mathrm{LO}(\pi_1(M_1)),$$

and set $N_2 = \mathrm{LO}(\pi_1(M_2))$, which one can check has only four elements. They are precisely the orderings that arise lexicographically from the short exact sequence

$$1 \to \langle y \rangle \to \langle x, y \mid xyx^{-1} = y^{-1} \rangle \to \mathbb{Z} \to 0.$$

There is a little bit of work to do here if we want to apply Bludov-Glass, where we need to check that

$$r(gP_D^{\pm\pm}g^{-1}) = r(P_D^{\pm\pm}),$$

where $r : \mathrm{LO}(\pi_1(M_1)) \to \mathrm{LO}(\partial M_1)$ is the restriction map. Once we check this, it is easy to see (e.g. by slogging through the definitions) that the two normal families are compatible with the identification induced by the gluing map. □

We had to fuss a bit in the example above to make the necessary normal families. However in some situations, sometimes the obvious necessary condition turns out to be enough. What we mean here by "obvious necessary condition" is the following: If $G *_{\phi_i} H$ is LO and so contains a positive cone P, then clearly $P \cap G = P_G$ and $P \cap H = P_H$ are two positive cones that satisfy $\phi_1^{-1}(P_G) = \phi_2^{-1}(P_H)$. So the existence of "compatible cones" P_G and P_H, in the sense that they agree on the amalgamated subgroups, is always necessary. In fact, it is sometimes sufficient.

Theorem 2.58. *Suppose that G, H, A are as above, that $\phi_1(A)$ is central in G and $\phi_2(A)$ is central in H. Then $G *_{\phi_i} H$ is LO if and only if there exist $P_G \subset G$ and $P_H \subset H$ with $\phi_1^{-1}(P_G) = \phi_2^{-1}(P_H)$.*

Proof. The "only if" part holds in general. On the other hand, suppose that there exist P_G, P_H as in the statement of the theorem.

Set

$$N_1 = \{P \in \mathrm{LO}(G) \mid \phi_1^{-1}(P) = \phi_1^{-1}(P_G)\}$$

and set

$$N_2 = \{P \in \mathrm{LO}(H) \mid \phi_2^{-1}(P) = \phi_2^{-1}(P_H)\}.$$

To see that these families are normal, note that if $P \in \mathrm{LO}(G)$ then

$$gPg^{-1} \cap \phi_1(A) = gPg^{-1} \cap g\phi_1(A)g^{-1} = g(P \cap \phi_1(A))g^{-1} = P \cap \phi_1(A)$$

since $\phi_1(A)$ is central in G. Therefore if $P \in N_1$ then $gPg^{-1} \cap \phi_1(A) = P \cap \phi_1(A)$ and therefore $\phi_1^{-1}(gPg^{-1}) = \phi_1^{-1}(P) = \phi_1^{-1}(P_G)$. Similarly for N_2.

Moreover, by these same observations, every $P \in N_1$ satisfies $\phi_1^{-1}(P) = \phi_2^{-1}(P_H)$, and every $Q \in N_2$ satisfies $\phi_1^{-1}(P_G) = \phi_2^{-1}(Q)$. So these families satisfy the hypothesis of Theorem 2.50, and thus $G *_{\phi_i} H$ is LO. □

We can generalize the "if" direction of the last theorem to prove things like:

Theorem 2.59. *Suppose that G, H, A are as above. If there exist $P_G \subset G$ and $P_H \subset H$ that are positive cones of bi-orderings with $\phi_1^{-1}(P_G) = \phi_2^{-1}(P_H)$, then $G *_{\phi_i} H$ is LO.*

There is also another extremely significant class of examples that are expected to behave this same way, in the sense that it's good enough to match left-orderings of the factors on the amalgamating subgroups, and the normal families somehow "happen for free".

Conjecture 2.60. [4] *Suppose that for $i = 1, 2$, the 3-manifold M_i is compact, connected, orientable and irreducible, with boundary $\partial M_i = T_i$ an incompressible torus. Fix a homeomorphism $\phi : T_1 \to T_2$ and set $M = M_1 \cup_\phi M_2$, whose fundamental group is $\pi_1(M) = \pi_1(M_1) *_{\phi_i} \pi_1(M_2)$ for some choice of injective homomorphisms $\phi_i : \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(M_i)$ determined by the gluing map ϕ (i.e. we want $\phi(\phi_1(a, b)) = \phi_2(a, b)$ for all $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$).*

Then $\pi_1(M)$ is LO if and only if there exist positive cones $P_1 \in \pi_1(M_1)$ and $P_2 \in \pi_1(M_2)$ such that $\phi_1^{-1}(P_1) = \phi_2^{-1}(P_2)$.

In fact, we conjecture something a bit different. First, observe that with M, M_1, M_2 and ϕ as above, the map ϕ induces an bijection $\phi_* : \mathcal{S}(M_1) \to \mathcal{S}(M_2)$ from the slopes on the boundary of M_1 to slopes on the boundary of M_2. Recall also that there is a slope map $s : \mathrm{LO}(\pi_1(M)) \to \mathcal{S}(M) \cong S^1$ which associates to each positive cone the equivalence class of a line $[L(P)]$.

Conjecture 2.61. [4] *With M, M_1, M_2 and ϕ as above, we use $s_i : \mathrm{LO}(\pi_1(M_i)) \to \mathcal{S}(M)$ to denote the slope map of the ordering. Then the fundamental group $\pi_1(M)$ is left-orderable if and only if $\phi_* \circ s_1(\mathrm{LO}(\pi_1(M_1))) \cap s_2(\mathrm{LO}(\pi_1(M_2))) \neq \emptyset$.*

In plain english: The fundamental group $\pi_1(M)$ is left-orderable if and only if we can find orderings on $\pi_1(M_1)$ and $\pi_1(M_2)$ that determine the same slope. This can be generalized to multiple 3-manifold pieces, but it's simplest if we stick with two for the statement of our conjecture.

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