Introduction to taut foliations on 3-manifolds I

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def^n: a $\underset{\text{codimension } (n-k)}{\underline{k\text{-dimensional}}}$ fol^n $\mathcal{F}$ of M^n is a decomposition of M into k-dim'l manifolds (called leaves)

$M = \bigcup_\alpha L_\alpha$

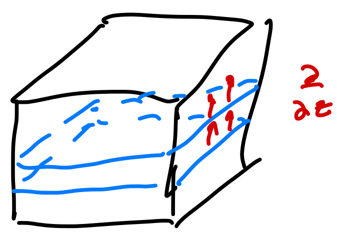
s.t. locally $(M, \mathcal{F})$ modeled by $(\mathbb{R}^n, \coprod_{z \in \mathbb{R}^{n-k}} \mathbb{R}^k \times \{z\})$

i.e. $\exists$ fol^n atlas on M

$$(U, U \cap \mathcal{F}) \approx (\mathbb{R}^n, \coprod \mathbb{R}^k \times \{z\})$$

$\underline{n=3}$: $k = 1$ or 2

locally



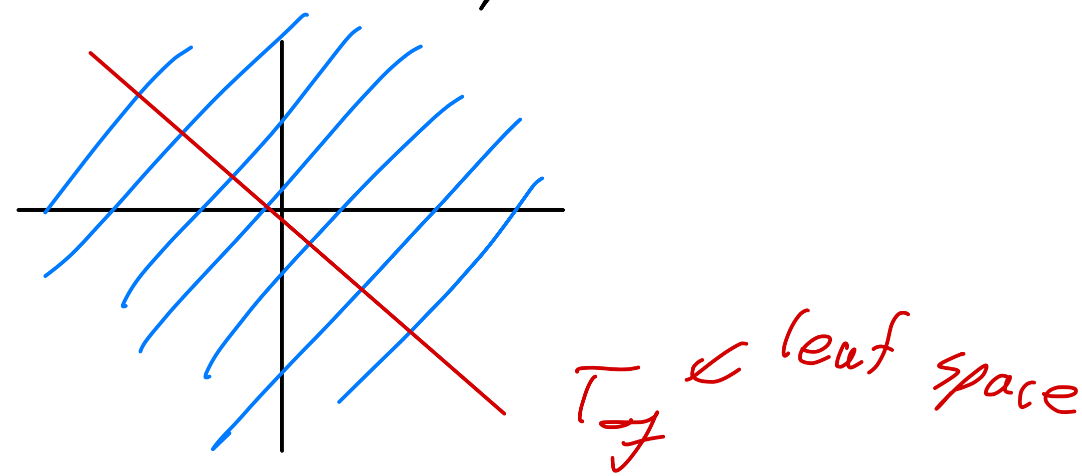
assume $\mathcal{F}$ is co-oriented or transversely oriented

surface leaves often non compact: injectively immersed

I. Developing some key ideas in the context $k=1$, $n=2$

eg $M=\mathbb{R}^2$

$\mathcal{F}$ = straight lines of same slope



leaf space of $\mathcal{F}$ = $M/\sim$ $\quad x \sim y$ if x, y on same leaf

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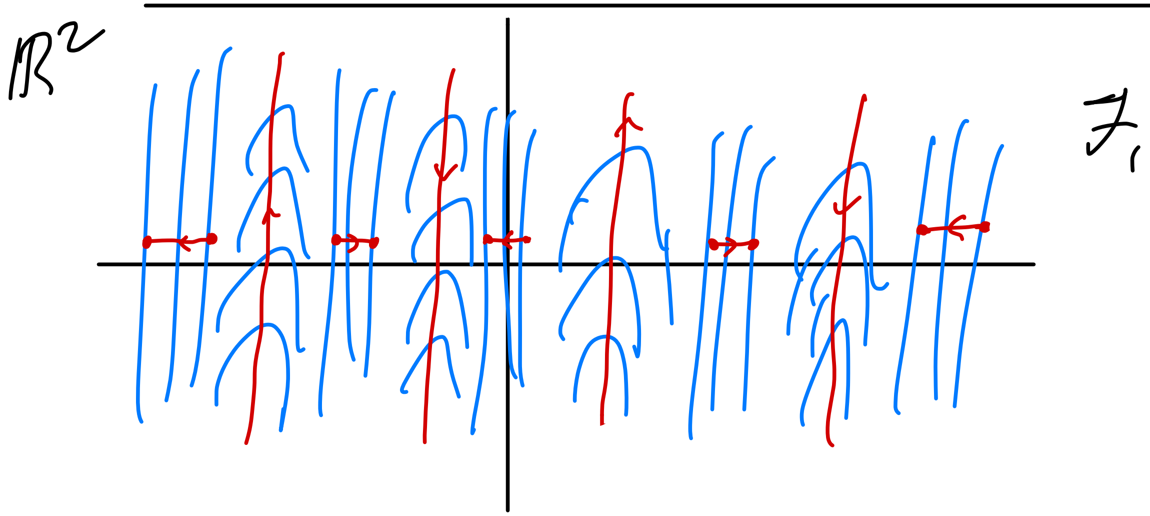
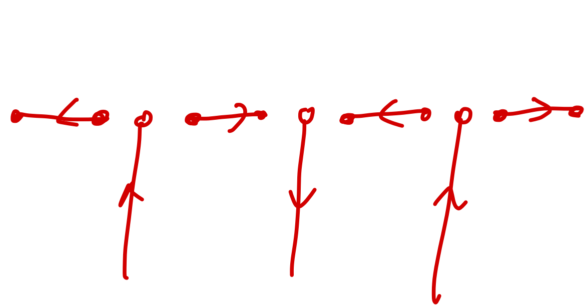
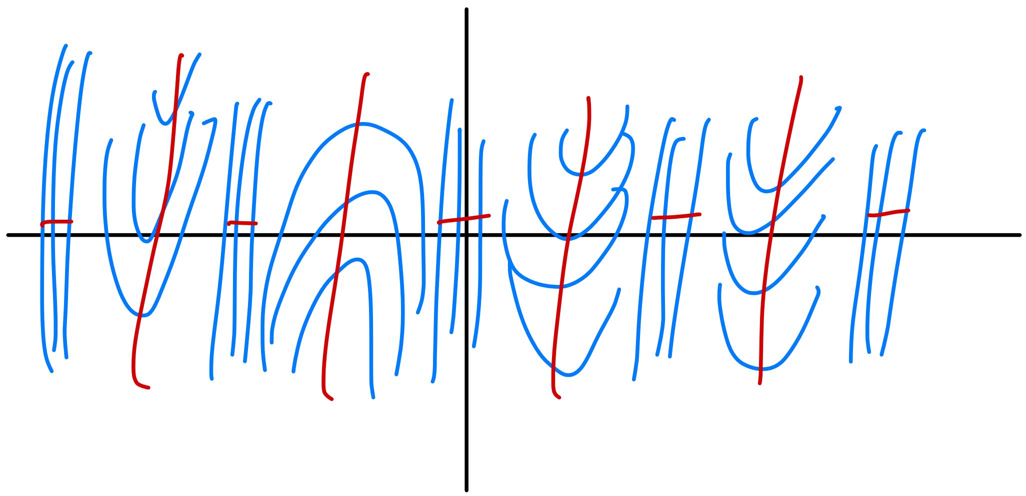




second countable
non-Hausdorff
1-manifold

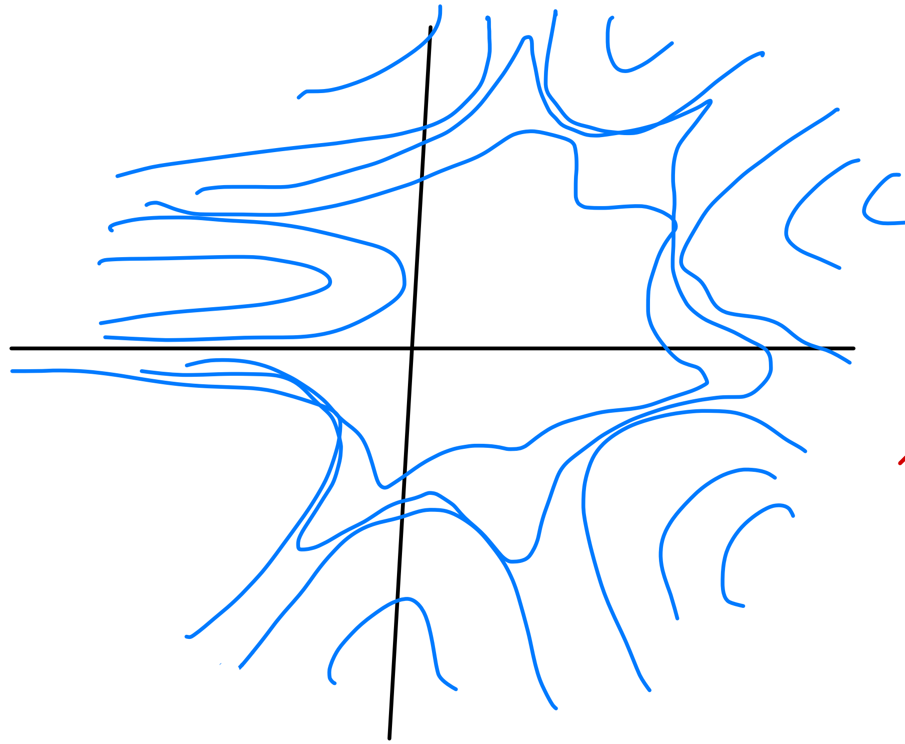


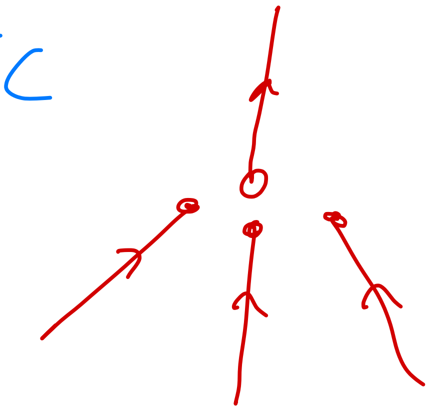
$\mathcal{F}_2$

<u>exercise</u>: same leaf space as above

<u>Remark</u>: $T_{\mathcal{F}_1} \cong T_{\mathcal{F}_2}$ but $\mathcal{F}_1$, $\mathcal{F}_2$ nonisotopic

eg:

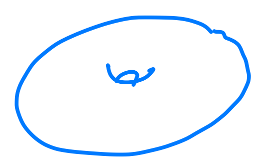




Poincaré-Hopf index theorem (Milnor's Topology from a differentiable viewpoint)

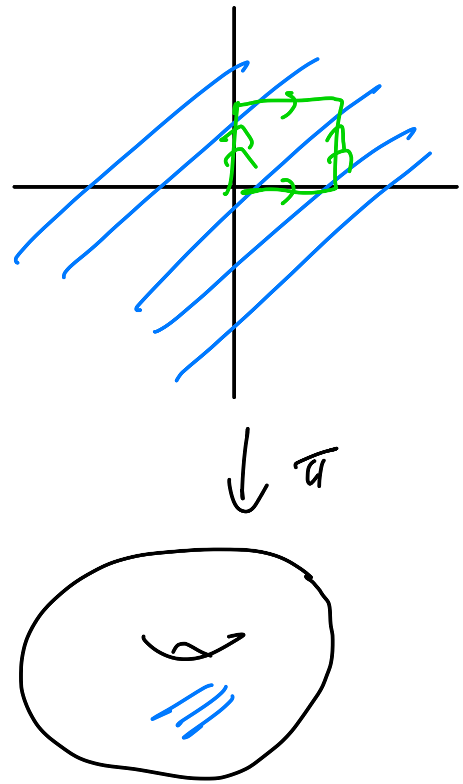
$\Rightarrow$ if M^2 closed has a codim 1 fol^n then

$$\chi(M) = 0$$

assume M orientable then $M =$ 

eg

$\tilde{\mathcal{F}}$



$\mathcal{F}$

$$(s,t) \longmapsto (e^{2\pi i s}, e^{2\pi i t})$$

<u>exercise</u> if m is slope of lines in $\mathbb{R}^2$

$m \in \mathbb{Q} \Rightarrow \pi(\tilde{\mathcal{F}}) =$ union parallel circles

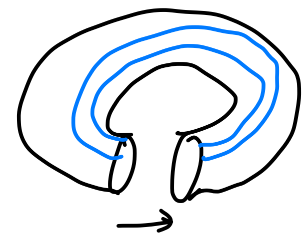
$m \notin \mathbb{Q} \Rightarrow \pi(\tilde{\mathcal{F}}) = \bar{L}$

$L =$ injective immersion of $\mathbb{R}$

every leaf = copy of $\mathbb{R}$

can also think of $T^2 \setminus S^1$ as

$A = [0,1] \times S^1$



glue by $f \in \mathrm{Homeo}^+(S^1)$

above $r \in \mathbb{Q}$ $\quad f =$ rational rotation

$r \notin \mathbb{Q}$ $\quad f =$ irrational rotation

$T_{\mathcal{F}}$ is complicated
(C^*-algebra)

$\{\dots f^{-1}(z_0), z_0, f(z_0), f^2(z_0), \dots\}$ dense in S^1

L is <u>dense</u> in M if $\bar{L} = M$

note in example above $\pi_1(M)$ acts on $\tilde{\mathcal{F}}$ in $\mathbb{R}^2$

$$\pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Classification of 1-dim'l fol$^{\text{n}}$s of T^2
(Hector–Hirsch)

dichotomy: either

(1) $\mathcal{F}$ contains a Reeb annulus

or

(2) $\mathcal{F}$ is a suspension lamination

i.e. comes from constr. above

for some $f \in \mathrm{Homeo}^+(S^1)$

Remark: $\mathcal{F}$ = fol^n of compact M then usually

$T_{\mathcal{F}}$ = leaf space of lift to universal cover

is more useful in understanding $\mathcal{F}$