Lecture 2

M^n $n \leq 3$ C^∞ structure
unique upto diffeo
(Hatcher: The Kirby torus trick for surfaces)

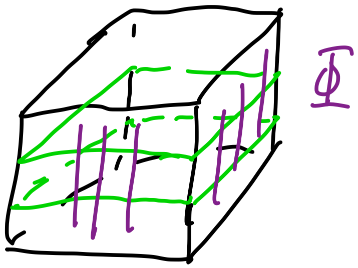
$\mathcal{F}$ = dim k in M^n
call $\mathcal{F}$ C^r if $\exists$ C^r foliated atlas

to rule out pathologies $T\mathcal{F}$ exists and is C^0

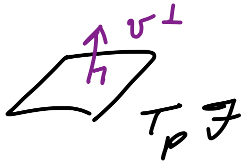
Hector-Hirsch ($n=2$)
Calegari ($n=3$)
} can always isotop $\mathcal{F}$ so that $T\mathcal{F}$ is C^0

leaves C^∞ immersed

Thm:
let $\mathcal{F}$ be a codim 1 co orientable fol^n of closed M s.t. $T\mathcal{F}$ is C^0
then $\exists$ transverse C^∞ 1-dim'l fol^n Φ



<u>Pf</u>: Fix Riemannian metric on M



$v^\perp$ be C^0-vector field perpendicular to $T\mathcal{F}$

approximate by C^∞ vector field ω

ω close enough to $v^\perp \Rightarrow$ non vanishing & $\pitchfork$ to $T\mathcal{F}$

Integrate to get Φ

eg: find Φ in examples from yesterday

An important operation

Denjoy blow-up

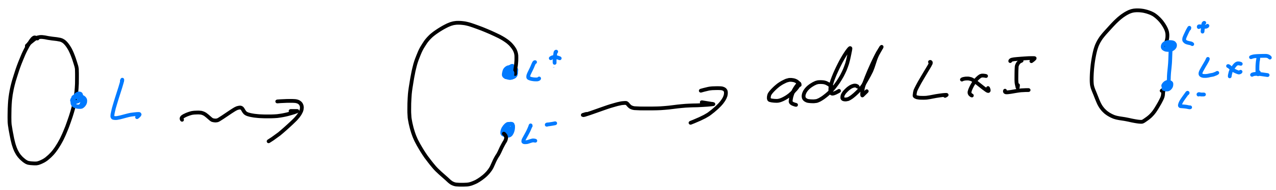
$\mathcal{F}$, L = leaf (or countably many leaves L_i)

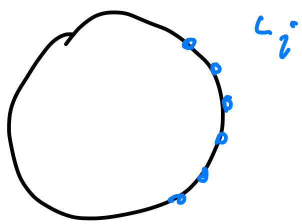
replace L_i with $L_i \times [0,1]$

to get a new foliation

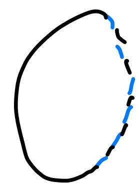
eg: $M^1 = S^1$

$$\mathcal{F} = \coprod_{\theta \in S^1} \{\theta\}$$





L_i = countable union of leaves

$S^1 \xrightarrow[\text{points}]{\text{remove}}$ 

add close intervals

exercise: you get circle back

$$S^1_{new} = \left(S^1 \times \{z_i\}\right) \cup \left(\bigcup_i J_i\right)$$

need $\sum \ell(J_i) < \infty$

Hint: see Cantor function

special case

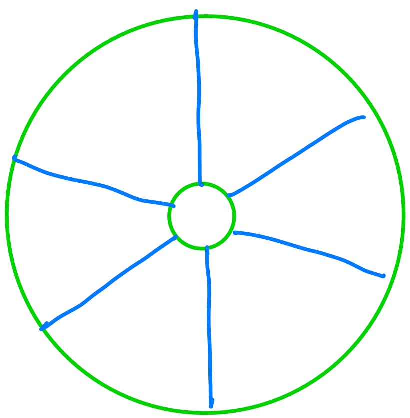
$$f : S^1 \to S^1 \qquad z_0 \in S^1$$

irrational rotation

$$\{z_i\} = \{f^i(z_0)\}_{i \in \mathbb{Z}}$$

Denjoy's Example

$\mathcal{F}$ = fol^n of T^2 given by suspending f = irrational rotation



$\{z_0\} \times [0,1]$

$$T = S^1 \times [0,1] \Big/ (x,0) \sim (f(x),1)$$

$$\mathcal{F} = \coprod \theta \times [0,1] \Big/ \sim$$

$$L = \mathrm{leaf}\left(\coprod f^n(x_0) \times [0,1] \big/ \sim \right)$$

replace by $L \times I$

$$\mathcal{F} \longrightarrow \mathcal{F}'$$

$$S'_{new} \times [0,1] \big/ (x,0) \sim (g(x),1)$$

on points $S'_{new} - \bigcup J_n$

$$g(z) = f(z)$$

on $J_n \times \{0\} \xrightarrow[\text{scaling}]{g \text{ is}} J_{n+1} \times \{0\}$

$$\mathcal{F}' = \coprod \theta \times [0,1] \big/ (x,0) \sim (g(x),1)$$

$S'_{new} - \bigcup \mathrm{int}\, J_n$ is a Cantor set

L leaf of $\mathcal{F}'$ NOT $L \times \{t\}$ $t \in [0,1]$

$$\Rightarrow L \not\subseteq T^2$$

def^n: a minimal set X of a foliation is a closed union of leaves that is minimal with respect to inclusion

equivalently, $\forall$ leaf L in X, $\bar{L} = X$

Thm:

let X = minimal set of codim 1 foliation then X has one of the following forms

(1) X = compact leaf

(2) $X = \mathcal{F}$

(3) X = transversely Cantor

called exceptional

Denjoy: $\mathcal{F}$ = fol^n of T^2 is $C^2 \Rightarrow \mathcal{F}$ does not contain an exceptional min^l set

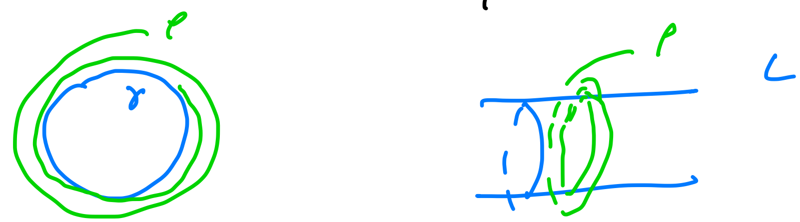
with care, the fol^n above is C^1

Sacksteder: an example of C^∞ fol^n $\mathcal{F}$ in M^3 that has an exceptional set

generalization of Denjoy (incomplete version)

$\mathcal{F} = C^2$ codim 1 fol^n that has an exceptional min'l set X has a resilient leaf (in X)

<u>def^n</u>: a leaf L of a codim 1 fol^n $\mathcal{F}$ is <u>resilient</u> if $\exists$ loop γ in L and ray $\rho : [0,\infty) \to L$ in L s.t. $\gamma \subset \bar{\rho}$



<u>exercise</u>: give an example of a codim 1 fol^n of a 3-mfd that has an exceptional min'l set without resilient leaf

<u>exercise</u>: given an example of a codim 1 fol^n of a 3-mfd that is not C^2 but has exceptional set w/ resilient leaf