Lecture 3

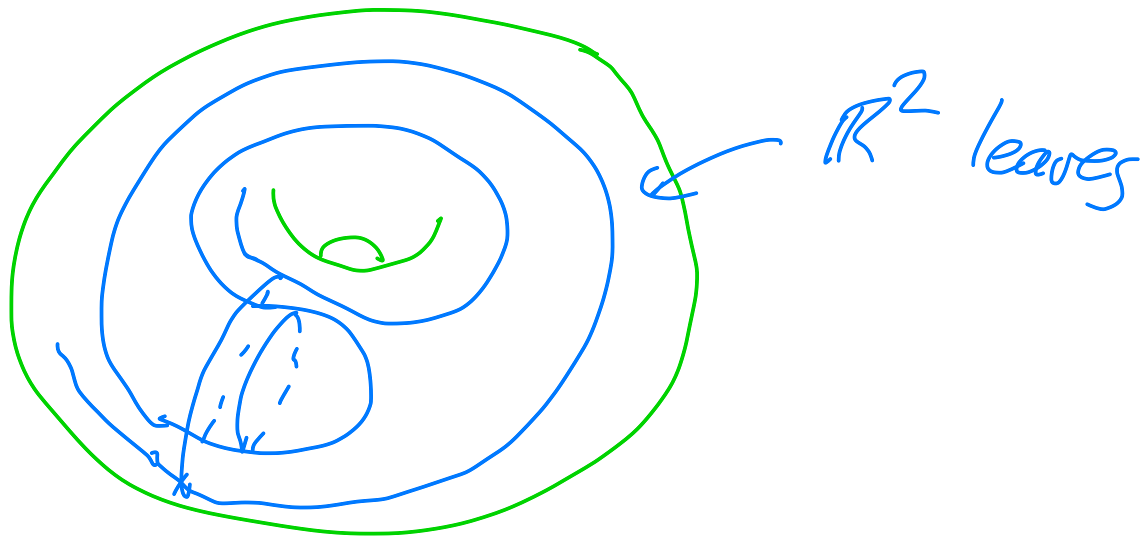
M^3 = compact orientable 3-mfd

$\mathcal{F}$ = codim 1 foliation co-oriented

example:

$M = S^1 \times D^2$

Reeb fol^n of $S^1 \times D^2$



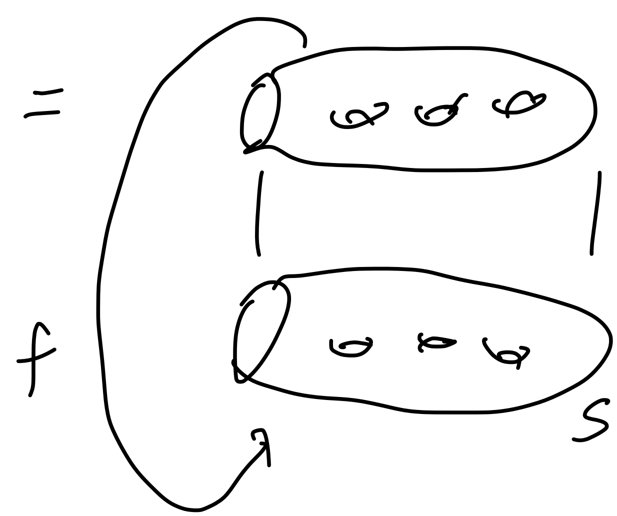
$\mathcal{F} = (S^1 \times \partial D^2) \cup$ circles worth of $\mathbb{R}^2$ leaves

$S^1 \times D^2$ with Reeb foliation = Reeb components

example: $S^3 = (S^1 \times D^2) \cup S^1 \times D^2$

$\mathcal{F}$ = Reeb fol$^{\underline{n}}$ $\cup$ Reeb fol$^{\underline{n}}$

Th$^{\underline{m}}$: any closed orientable 3-mfd is a Deh$^{\underline{n}}$ filling of fibered over S^1 3-mfd with single ∂-component

M =  $\cup_\partial (S^1 \times D^2)$

$f \in Homeo^+(S)$

Cor: every closed orientable 3-manifold admits a codim 1 trans. orientable fol^n

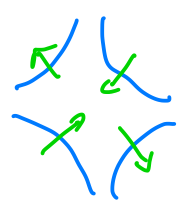
Put Reeb of $S^1 \times D^2$ above then "spiral" the S-fibers above to $\partial(S^1 \times D^2)$

examples:

1) (any fol^n of T^2) $\times S^1$ fol^n of T^3

2) $M = S \times [0,1] / (x,1) \sim (f(x), 0)$ f = pseudo-Anosov map

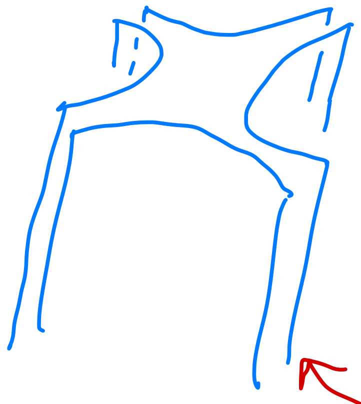
λ = stable lamination (assume λ oriented and f preserves or^n)



$\Lambda = \lambda \times [0,1] / (x,1) \sim (f(x), 0)$

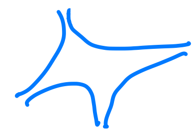
= minimal set of a fol^{n} $\mathcal{F}$

get $\mathcal{F}$ by "stacking chairs"



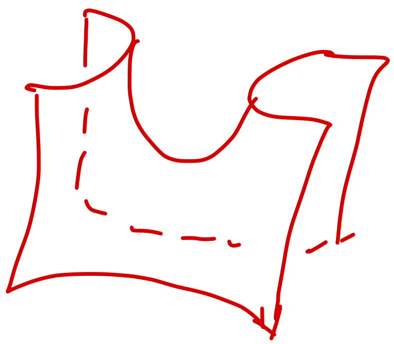
each complementary region $M - \Lambda$
= cusped solid torus

cusped-gon $\times I/\sim$



$\times I$

stack of these in



this gives a Reebless fol^{n}

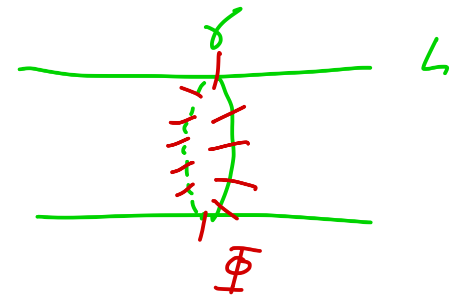
Holonomy and Reeb stability

def^{n}

L = leaf of $\mathcal{F}$

$p \in L$

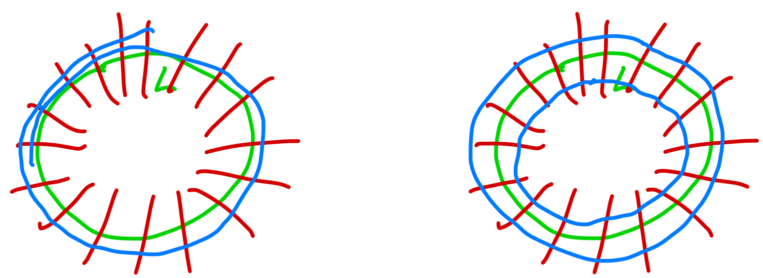
γ loop in L based at p



Φ = dim 1 fol^{n} $\pitchfork$ $\mathcal{F}$

take union of little intervals of Φ along γ

called a <u>normal fence</u> along γ



by following leaves of $\mathcal{F}$ along normal fence

we obtain a map

$f_\gamma : \tau \to \tau$ { leaf of normal fence through $p \in \gamma$

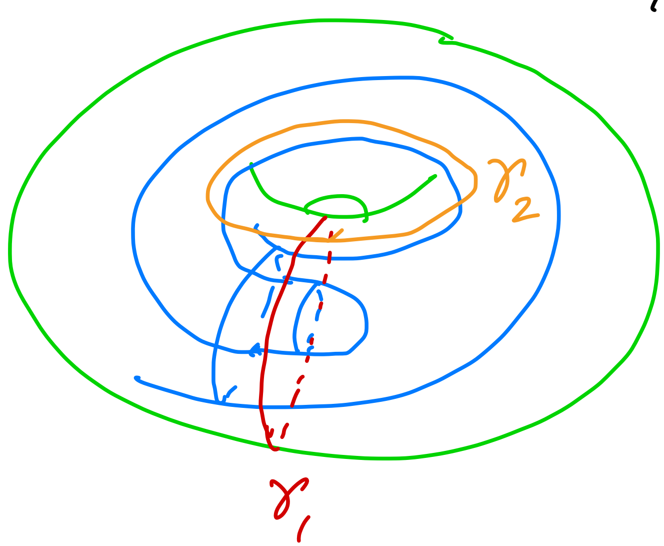
the map – more precisely the germ – is

called the <u>holonomy of $\mathcal{F}$ along γ</u>

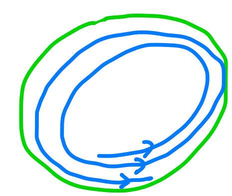
Say the holonomy of γ is trivial if $f_\gamma = id$ on some open interval of τ about p

<u>Fact:</u> f_γ is determined by $[\gamma]$ (homotopy class γ in L)

<u>example:</u>



trivial holonomy about γ_1
nontrivial holonomy about γ_2



say $\mathcal{F}$ has trivial holonomy if $\forall$ leaf L of $\mathcal{F}$ and $\forall \gamma$ in L, $f_\gamma = id$

Reeb stability thm: given $(M, \mathcal{F})$

L = compact leaf with trivial holonomy

$\Rightarrow$ L has a nbhd $\cong L \times I$ s.t.

$$\mathcal{F}\big|_{L \times I} = \underset{\text{product fol}^{\text{n}}}{L \times I}$$

Cor: if $\mathcal{F}$ contains a leaf $\approx S^2$ then

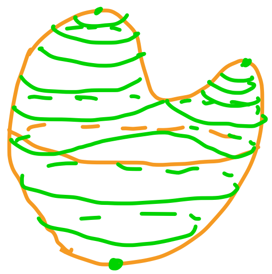
$$M = S^1 \times S^2 \ \& \ \mathcal{F} = \coprod (\{\theta\} \times S^2)$$

Idea: given $(M, \mathcal{F})$

understand submanifold X in M
by isotoping them to lie in general
position with respect to leaves of $\mathcal{F}$
(aka: allows us to play Morse theory games)

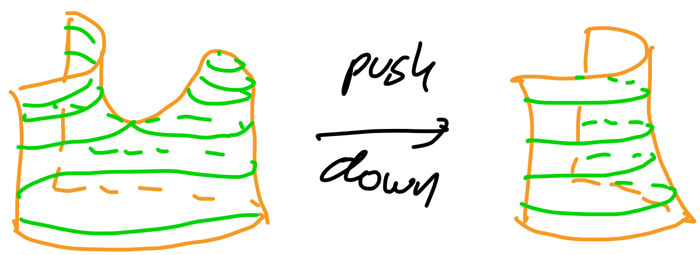
example: special case

$\mathbb{R}^3 \quad \mathcal{F} = \coprod_{z \in \mathbb{R}} \mathbb{R}^2 \times \{z\}$



$S^2 = X$

isotop to minimize critical points



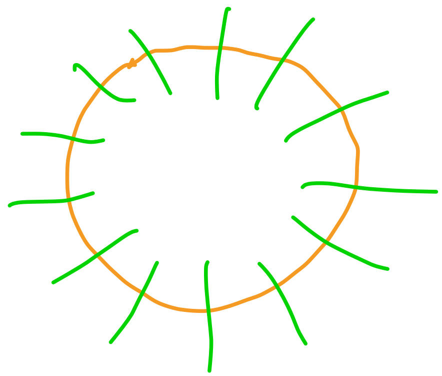
Question: Given M, γ = loop in M

what can you say about $[\gamma] \in \pi_1(M)$

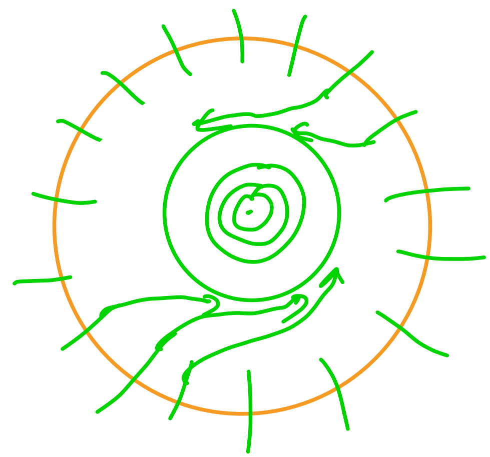
Special Case: Can find $\mathcal{F}$ s.t. γ can be isotoped s.t. $\gamma \pitchfork \mathcal{F}$

Claim: In this case, what can we say?

example: assume γ embedded and bounds embedded disk D $(=X)$



Poincare-Hopf must have a center



So in local chart

Reeb fol^{n} is example of this

Novikov vanishing cycle $= \mathcal{F} \cap D^2$ that has the form

$\gamma_t \quad 0 \le t \le 1$

γ_t bounds a disk in its leaf
$\forall t < 1$ but **not** when $t = 1$

Thm: vanishing cycle $\Rightarrow \exists$ Reeb component

Fact: no spiralling around trivial loop

Cor (Novikov):

Reebless $\mathcal{F}$
if $\gamma \pitchfork \mathcal{F}$ then $[\gamma]$ = infinite order in $\pi_1(M)$

Novikov:

Suppose $M \neq S^1 \times S^2$, $\mathcal{F}$ = Reebless
(1) $\gamma \pitchfork \mathcal{F} \Rightarrow [\gamma]$ inf order
(2) leaves of $\mathcal{F}$ are π_1-injective
(3) $\pi_2(M) = 0$ (intersection $\mathcal{F} \cap f(S^2)$)

Rosenberg:

M contains Reebless fol$^{\underline{n}}$

then M irreducible