Lecture 4

M orientable 3-manifold
$\mathcal{F}$ trans. orient codim 1

(Rosenberg-Sondow, Tao Li)

$(M, \mathcal{F})$ all leaves planes $\Rightarrow M = T^3$

(Palmeira)

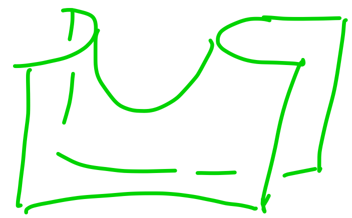
$(M, \mathcal{F})$ $\mathcal{F}$ Reebless $\Rightarrow \tilde{M} = \mathbb{R}^3$

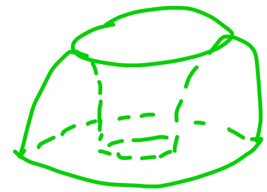
"Pf": $\mathcal{F}$ Reebless $\Rightarrow$ leaf space $\tilde{\mathcal{F}}$ on univ. cover
is simply connected 2^{nd} countable,
not nec. Hausdorff 1-manifold
roughly speaking, tree = countable union
$\Rightarrow \tilde{M}$ nested union of 3-mfd ...

(Roussarie-Thurston)

$(M, \mathcal{F} = \text{Reebless})$
$S \hookrightarrow M$ π_1-injective closed surface $g \geq 1$
$\Rightarrow S$ can be isotoped so that $S \pitchfork \mathcal{F}$
except at a finite # of saddle

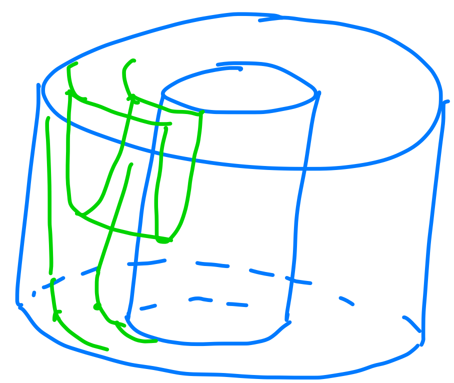
and circle tangencies





as long as no cylindrical components, can remove circle tangencies

Reeb annulus $\times S^1$

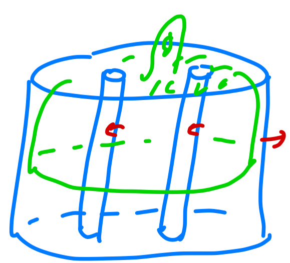


Thurston:

orientable compact surface F, $\partial F \neq \emptyset$
and fol^n contains as leaves

$F \times I/\sim$

spiral to torus leaf



key: torus leaves co-oriented so all point in or all point out

def^n (Sue Goodman):

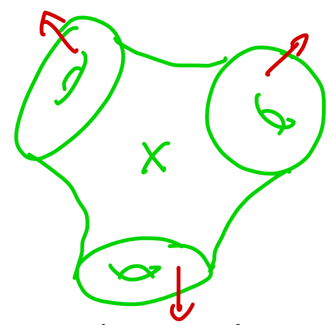
Given $(M^3, \mathcal{F})$
↖ oriented

a <u>dead end component</u> of $\mathcal{F}$ is a sub mfd X (oriented) of M

$\partial X = \coprod T^2 \neq \emptyset$

$\mathcal{F}|_X$ = fol^n tangent to ∂X

co-orientation agrees with $\pm$ orientation of ∂X induced by or^n on X



once you flow into X by ⋔ 1-dim flow never flow out

Fact: given $(M, \mathcal{F})$
no dead end components
$\Longleftrightarrow$
$\mathcal{F}$ taut ($\exists$ loop γ in M s.t. $\gamma \pitchfork \mathcal{F}$ and sees every leaf of $\mathcal{F}$)

Thurston '76:

M^3 compact orientable

define Thurston norm on $H_2(M)$

$$\|z\| = \min\{-\chi(S') \mid [S]=z \text{ and } S' = S - \text{any } S^2 \text{ components}\}$$

Th^m (Thurston):

$(M, \mathcal{F})$, L = compact leaf of $\mathcal{F}$

$$-\chi(L) = \|[L]\|$$

(minimal genus in its homology class)

Sketch of Pf:

$[F] = [S]$

put F in good form w.r.t. $\mathcal{F}$

use Poincaré-Hopf to show

$$-\chi(S) \leq -\chi(F)$$

(Gabai-Yazdi) M hyperbolic

$\mathcal{F}$ taut

suppose S is fully marked by $\mathcal{F}$

i.e. saddle tangencies all of same sign

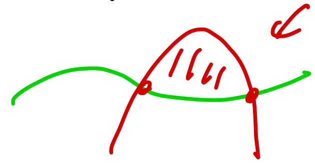
($\therefore$ minimal genus)

can find a surface F homologous to S
and a taut fol^n $\mathcal{G}$ s.t. F is a leaf of $\mathcal{F}$
and $T\mathcal{F}$ and $T\mathcal{G}$ are homotopic

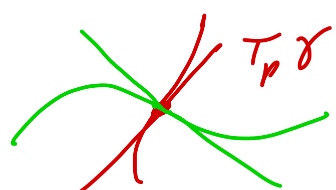
Tautness (C^2-Sullivan)

- $\gamma \pitchfork_{top} \mathcal{F}$ means no non-degenerate subarc of γ isotopes into a leaf of $\mathcal{F}$

  ie don't see



- γ C^∞ loop
  $\gamma \pitchfork_\infty \mathcal{F}$ means $T_p\gamma \oplus T_p\mathcal{F} = T_pM$



- $\mathcal{F}$ is <u>everywhere taut</u> if $\forall p \in M$, $\exists\, \gamma \pitchfork_{\infty} \mathcal{F}$ such that $p \in \gamma$

<u>Fact</u> (Colin-Kazez-R)

every top taut $\mathcal{F}$ can be isotoped to be everywhere taut

<u>Issue:</u> $T\mathcal{F}$ only C^0, $\mathcal{F}$ top taut

then not nec. uniquely integrable

so there migh exist $\mathcal{G}$ fol^n

s.t. $T\mathcal{G} = T\mathcal{F}$ and $\mathcal{G}$ contains

deadend components

<u>Th^m</u> (Eliashberg-Thurston)

M closed, oriented $\neq S^1 \times S^2$

any taut C^2 fol^n on M can be C^0-approximated

by both a positive and a negative contact

structure $\xi_{\pm}$ and they are universally tight

and weakly symplectically fillable

Cor (Ozsváth-Szabó)

M an L-space $\Rightarrow$ M cannot contain a taut fol^{n}

Contrast with

Th^{m} (Bowden, Kazez-R)

M closed oriented $\neq S^1 \times S^2$ any trans. orient. with $T\mathcal{F}$ C^0 can be approx by $\xi_\pm$

Th^{m} (Colin-Kazez-R)

$\exists$ $\mathcal{F}$ top taut that can be C^0-approx by both $\xi_\pm$ weakly symp fillable, universally tight and $\xi'_\pm$ overtwisted

if everywhere taut $\xi_\pm$ nec. weak symp fill and univ. tight

Th^{m} (C^0 version of Sullivan)

M = closed, conn, oriented

$\mathcal{F}$ trans orient C^0

$\mathcal{F}$ everywhere taut

$\Longleftrightarrow$

(1) $\exists$ C^∞ closed 2-form ω on M s.t. $\omega > 0$ on $T\mathcal{F}$

$\Longleftrightarrow$

(2) $\forall$ Riem metrics on M, $\exists$ a volume pres. transverse flow

Sullivan: C^2 taut $\mathcal{F}$

$\Longleftrightarrow$

geometrically taut

($\exists$ metric M s.t. every leaf is a minimal surface)

Question: What's the generalization to C^0 case?