(An algorithm for <u>not</u>) biordering links via braids

joint work in progress w/ J. Johnson & N. Scherich

Why biorder (3-mfd) group?

$\pi_1(S^3 - L) \cong \pi_1(S^3 - \nu(L)) =: \pi_1(L)$

$$\pi_1(L) \xrightarrow{\text{abel.}} H_1(S^3 - L; \mathbb{Z}) \twoheadrightarrow \mathbb{Z}$$
$$H_1(S^3 - L; \mathbb{Z}) \cong \mathbb{Z}^n$$

(Boyer-Rolfsen-Wiest) $\Rightarrow$ $\pi_1(L)$ is LO

<u>Question</u>: When is $\pi_1(L)$ BO?

<u>def^n</u>: G is <u>biorderable</u> if there is a strict total order on G invariant under both right and left multiplication

<u>notation</u>: L is BO if $\pi_1(L)$ is BO

<u>Th$^{\underline{m}}$ (Clay–Rolfsen)</u>:

if K is BO knot, then K is not an L-space knot

Remark: there are BO links which are L-space links

Problem: given K consider $S_K = \{ \Sigma_n(K) \}_{n=2}^{\infty}$

$\Sigma_n(K)$ cyclic n-fold branch cover

determine K so that S_K consists entirely of L-spaces

① How to show G is BO? ⟶ find one strict total order which is invar under right, left mult.

② How to show G is not BO? ⟶ no strict total order exists which is invar under rt, lt mult

Algorithm: If a braided link L is not BO, then our program returns "NO" and a proof that L is not BO.

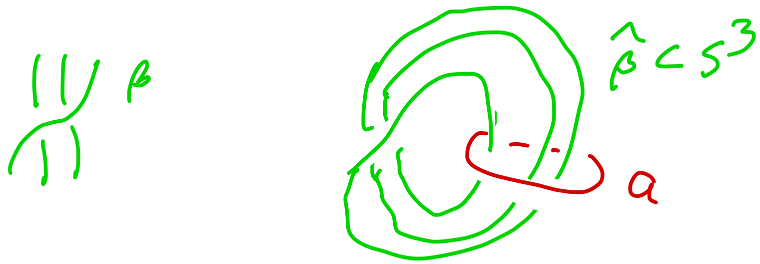
If L is BO, the program does not terminate

[Calagari - Dunfield]

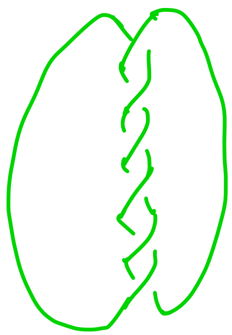
Braided links

a braided link is a link $L = \beta \cup a$ where

β is an n-strand braid and a is the braid axis of β



example:

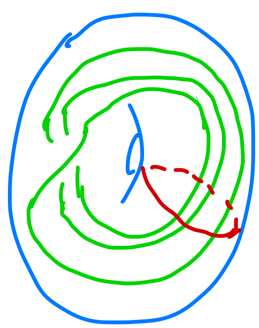


is braided

the complement of L is fibered

$$S^3 - \nu(a) - \hat{\beta} \text{ is fibered}$$

$$\underset{S^1 \times D^2}{\overset{\shortparallel}{S^3 - \nu(a)}} \cong$$



$M \cong S^3 - \nu(a) - \hat{\beta}$ is fibered D_n over S^1

disk w/ n punctures

$$1 \to \pi_1(D_n) \xrightarrow{i} \pi_1(M) \to \pi_1(S^1) \to 0$$

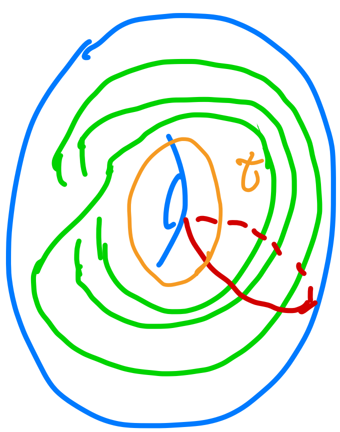
$\pi_1(D_n) \cong F_n$: LO, BO

$\pi_1(S^1) \cong \mathbb{Z}$: LO, BO

so $\pi_1(M)$ is LO

$\pi_1(M)$ is BO iff there is a conjugation invariant positive cone P of $\pi_1(D_n)$ so that $i(P)$ is conj-invariant in $\pi_1(M)$

in this case only need to check conjugation by t



this is determined by the action β on $\pi_1(D_n)$

$$t x t^{-1} = \beta_*(x)$$

Def^n: a braid is <u>order-preserving</u> (OP) if there is a biordering of F_n which β leaves invariant

i.e. whenever $x < y$ then $\beta(x) < \beta(y)$

Th^m (Kin-Rolfsen):

A braided link $L = \hat{\beta} \cup a$ is BO

iff

β is order-preserving

Kin-Rolfsen results:

- pure braids are OP
- periodic braids
- $\sigma_1 \sigma_2^{-1}$ not OP and pseudo Anosov $\sigma_1^{2} \sigma_2^{-1}$

Algorithm:

Build all positive cones of F_n which are conj invariant and β-invariant and

$$\mathrm{OP}\left\{\begin{array}{l}\mathrm{BO}\left\{\begin{array}{l}\mathrm{LO}\left\{\begin{array}{ll}(1) & P\cup P^{-1}\cup\{id\}=F_n \\ (2) & PP\subset P\end{array}\right.\\ (3)\quad gPg^{-1}\subset P\end{array}\right.\\ (*)\quad \beta(P)\subset P\end{array}\right.$$

(STO) $f<g \Leftrightarrow fg^{-1}\in P$

restrict to words of length at most k

Def: $P_k \subset F_n$ is a (conj, β-invt) pre-cone of length k if

(1) $P_k \cup P_k^{-1} \cup \{id\} = W_k$ words in F_n of length at most k

(2) $(P_k \cdot P_k) \cap W_k \subset P_k$

(3) $(g P_k g^{-1}) \cap W_k \subset P_k$

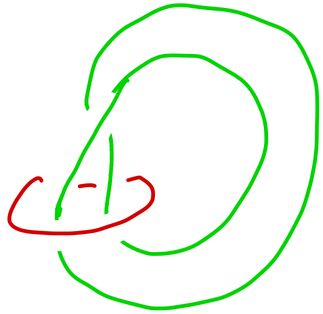
(*) $\beta(P_k) \subset P_k$

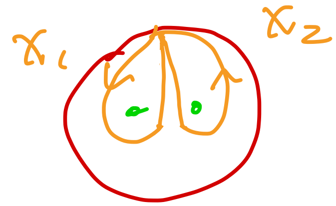
Prop: if P is a pos. cone invt under β, conj, then $P \cap W_k = P_k$ is a pre-cone of length k, $k \in \mathbb{Z}$

Find k so that no pre-cone exists

example:

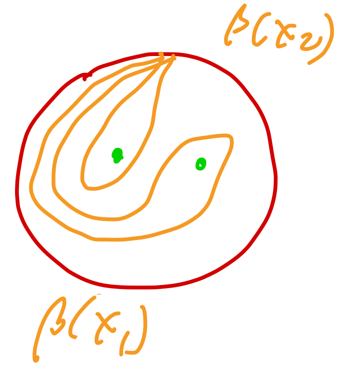
$\beta = \sigma_1$





want to show no biorder of F_2 is preserved by β

suppose $x_1 < x_2$ $(x_2 < x_1)$



$x_1 < x_2$

$\Rightarrow \beta(x_1) < \beta(x_2)$

$x_1 x_2 x^{-1} < x_1$

$\Rightarrow x_2 < x_1$ $\otimes$

$\beta(x_1) = x_1 x_2 x_1^{-1}$

$\beta(x_2) = x_1$

Computer

Start trying to build precone of length k

$k = 3$ without loss of gen $x_1 \in P$

(1) add all conjugates of things already in P

(2) add all images of things in P under β to P

(3) add all products of things in P

$k=3$ add x_1

1st time around

(1) $x_2 x_1 x_2^{-1}$, $x_2^{-1} x_1 x_2$

(2) $x_1 x_2 x^{-1}$

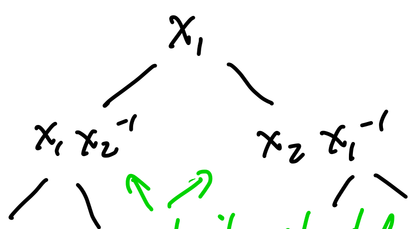
(3) x_1^2

2nd time around

(1) x_2

(2) ...

(3) ...



which should be in P_k?

if we put id in P_k then $\otimes$

eg if add $x_2 x_1^{-1}$ to P_k

(1) also has $x_1^{-1} x_2$ (conjugate of $x_2 x_1^{-1}$)

(2) $\beta(x_1^{-1} x_2) = x_1 x_2^{-1} x_1^{-1} x_1$
$= x_1 x_2^{-1}$

(3) $\therefore x_2 x_1^{-1} x_1 x_2^{-1} = \mathrm{id}$ $\otimes$

Improvements to the Algorithm

<u>Prop</u>: if β preserves some conj-invt cone of F_n then it preserves one where all elements with positive exponents sum are in P