A unified Casson-Lin invariant for the real forms of SL(2)

Nathan Dunfield (University of Illinois)

Joint with Jake Rasmussen

Based on arXiv:2209.03382

Notes already posted at:
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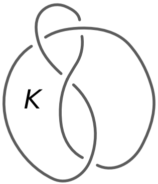

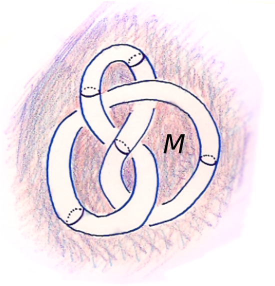

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Cor. If M is small with σ_K non-constant, then there is an irred $\rho\colon \pi_1 M \to \mathrm{SL}_2\mathbb{R}$.

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[Kronheimer-Mrowka] A nontrivial K has an irred $\rho\colon \pi_1 M \to \mathrm{SU}_2$.

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Motivation: L-space conjecture, orderability of 3-manifold groups, translation extension locus [Culler-D].

Let $\Sigma_n(K)$ be the n-fold cyclic cover of S^3 branched over K.

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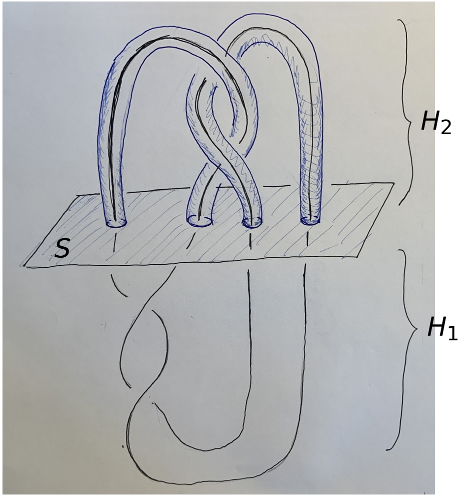
Casson-Lin signs. Write $M = H_1 \cup_S H_2$ using an n-bridge diagram for K:



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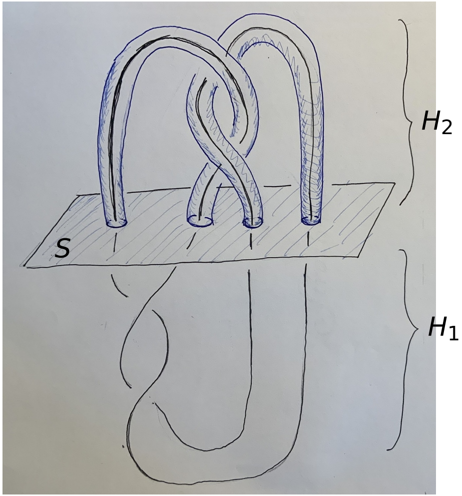

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$$\begin{array}{ccccc} & & \pi_1 H_2 & & \\ & \swarrow & & \nwarrow & \\ \pi_1 M & & & & \pi_1 S \\ & \nwarrow & & \swarrow & \\ & & \pi_1 H_1 & & \end{array}$$

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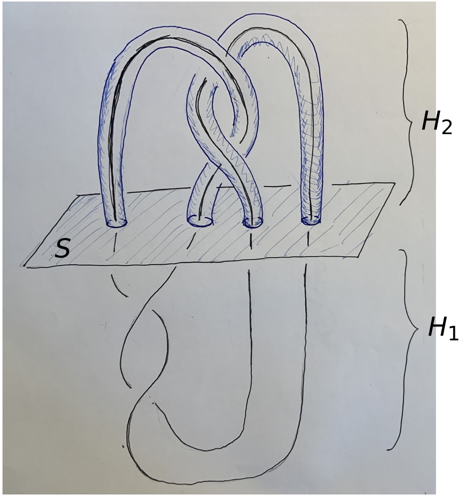

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Important: Even for $G = \mathrm{SU}_2$, these manifolds are all noncpt. But $X_G^{\theta,\mathrm{irr}}(M)$ is cpt when $\theta \notin D_M$ and M small.

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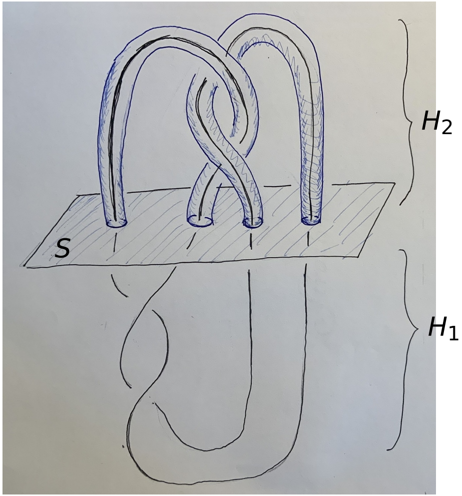

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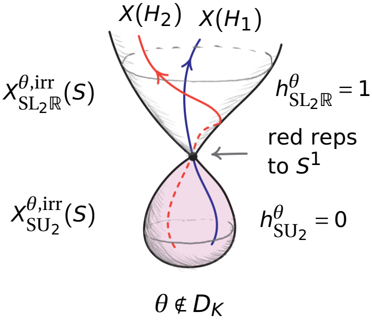

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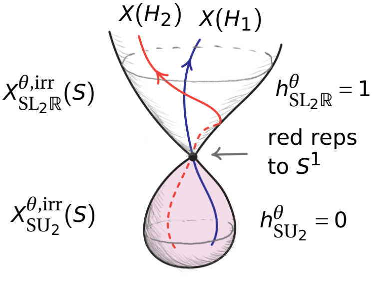

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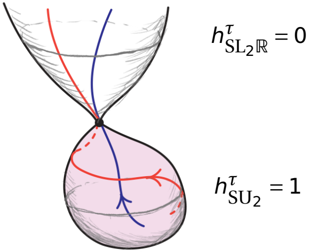

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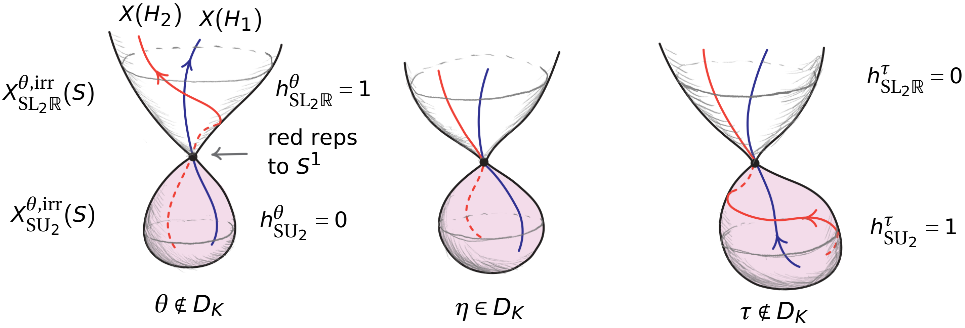

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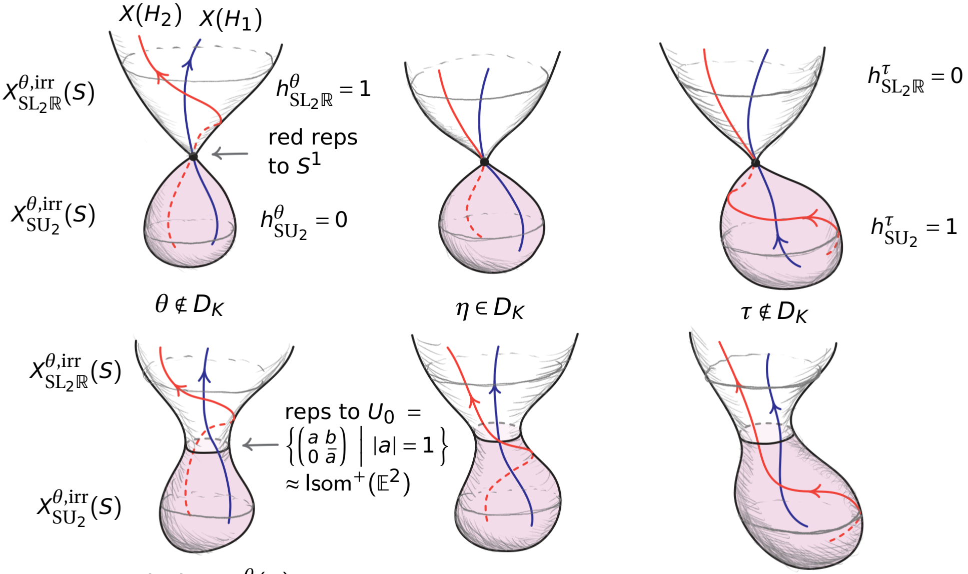

Resolution $\mathscr{X}^{\theta}(S)$

Moral: in resolved picture $h(M)$ is the alg $\cap\#$ of red and blue for **all** angles.